(3) Each element of an Abelian group forms a conjugacy class by itself.
(4) Elements of the same conjugacy class have the same order.
(5) Different conjugacy classes may contain different numbers of elements, *i.e.* have different 'lengths'.

Not only the individual elements of a group $\mathcal{G}$ but also the subgroups of $\mathcal{G}$ can be classified in conjugacy classes.

Definition 1.2.4.3.2. Two subgroups $\mathcal{H}_j, \mathcal{H}_k < \mathcal{G}$ are called *conjugate* if there is an element $g_q \in \mathcal{G}$ such that $g_q^{-1}\mathcal{H}_j\, g_q = \mathcal{H}_k$ holds. This relation is often written $\mathcal{H}_j^{g_q} = \mathcal{H}_k$. □

Remarks:

(1) The 'trivial subgroup' $\mathcal{I}$ (consisting only of the unit element of $\mathcal{G}$) and the group $\mathcal{G}$ itself each form a conjugacy class by themselves.
(2) Each subgroup of an Abelian group forms a conjugacy class by itself.
(3) Subgroups in the same conjugacy class are isomorphic and thus have the same order.
(4) Different conjugacy classes of subgroups may contain different numbers of subgroups, *i.e.* have different lengths.

Equation (1.2.4.1) can be written

$$\mathcal{H} = g_p^{-1}\mathcal{H}\, g_p \text{ or } \mathcal{H} = \mathcal{H}^{g_p} \text{ for all } p;\ 1 \le p \le i. \qquad (1.2.4.2)$$

Using conjugation, Definition 1.2.4.2.3 can be formulated as

Definition 1.2.4.3.3. A subgroup $\mathcal{H}$ of a group $\mathcal{G}$ is a normal subgroup $\mathcal{H} \lhd \mathcal{G}$ if it is identical with all of its conjugates, *i.e.* if its conjugacy class consists of the one subgroup $\mathcal{H}$ only. □

1.2.4.4. *Factor groups and homomorphism*

For the following definition, the 'product of sets of group elements' will be used:

Definition 1.2.4.4.1. Let $\mathcal{G}$ be a group and $\mathcal{K}_j = \{g_{j_1}, \ldots, g_{j_n}\}$, $\mathcal{K}_k = \{g_{k_1}, \ldots, g_{k_m}\}$ be two arbitrary sets of its elements which are not necessarily groups themselves. Then the product $\mathcal{K}_j\,\mathcal{K}_k$ of $\mathcal{K}_j$ and $\mathcal{K}_k$ is the set of all products $\mathcal{K}_j\,\mathcal{K}_k = \{g_{j_p}\, g_{k_q} \mid g_{j_p} \in \mathcal{K}_j,\ g_{k_q} \in \mathcal{K}_k\}$.[4] □

The coset decomposition of a group $\mathcal{G}$ relative to a normal subgroup $\mathcal{H} \lhd \mathcal{G}$ has a property which makes it particularly useful for displaying the structure of a group.

Consider the coset decomposition with the cosets $\mathcal{S}_j$ and $\mathcal{S}_k$ of a group $\mathcal{G}$ relative to its subgroup $\mathcal{H} < \mathcal{G}$. In general the product $\mathcal{S}_j\,\mathcal{S}_k$ of two cosets, *cf.* Definition 1.2.4.4.1, will not be a coset again. However, if and only if $\mathcal{H} \lhd \mathcal{G}$ is a normal subgroup of $\mathcal{G}$, the product of two cosets is always another coset. This means that for the set of all cosets of a normal subgroup $\mathcal{H} \lhd \mathcal{G}$ there exists a law of composition for which the closure is fulfilled. One can show that the other group postulates are also fulfilled for the cosets and their multiplication if $\mathcal{H} \lhd \mathcal{G}$ holds: there is a neutral element (which is $\mathcal{H}$), for each coset $g\,\mathcal{H} = \mathcal{H}\,g$ the coset $g^{-1}\,\mathcal{H} = \mathcal{H}\,g^{-1}$ forms the inverse element and for the coset multiplication the associative law holds.

Definition 1.2.4.4.2. Let $\mathcal{H} \lhd \mathcal{G}$. The cosets of the decomposition of the group $\mathcal{G}$ relative to the normal subgroup $\mathcal{H} \lhd \mathcal{G}$ form a group with respect to the composition law of coset multiplication. This group is called the *factor group* $\mathcal{G}/\mathcal{H}$. Its order is $|\mathcal{G} : \mathcal{H}|$, *i.e.* the index of $\mathcal{H}$ in $\mathcal{G}$. □

A factor group $\mathcal{F} = \mathcal{G}/\mathcal{H}$ is not necessarily isomorphic to a subgroup $\mathcal{H}_j < \mathcal{G}$.

Factor groups are indispensable for an understanding of the homomorphism of one group onto the other. The relations between a group $\mathcal{G}$ and its homomorphic image are very strong and are expressed by the following lemma:

Lemma 1.2.4.4.3. Let $\mathcal{G}'$ be a homomorphic image of the group $\mathcal{G}$. Then the set of all elements of $\mathcal{G}$ that are mapped onto the unit element $e' \in \mathcal{G}'$ forms a normal subgroup $\mathcal{X}$ of $\mathcal{G}$. The group $\mathcal{G}'$ is isomorphic to the factor group $\mathcal{G}/\mathcal{X}$ and the cosets of $\mathcal{X}$ in $\mathcal{G}$ are mapped onto the elements of $\mathcal{G}'$. The normal subgroup $\mathcal{X}$ is called the *kernel* of the mapping; it forms the unit element of the factor group $\mathcal{G}/\mathcal{X}$. A homomorphic image of $\mathcal{G}$ exists for any normal subgroup of $\mathcal{G}$. □

The most important homomorphism in crystallography is the relation between a space group $\mathcal{G}$ and its homomorphic image, the point group $\mathcal{P}$, where the kernel is the subgroup $\mathcal{T}(\mathcal{G})$ of all translations of $\mathcal{G}$, *cf.* Section 1.2.5.4.

1.2.4.5. *Normalizers*

The concept of the normalizer $\mathcal{N}_\mathcal{G}(\mathcal{H})$ of a group $\mathcal{H} < \mathcal{G}$ in a group $\mathcal{G}$ is very useful for the considerations of the following sections. The size of the conjugacy class of $\mathcal{H}$ in $\mathcal{G}$ is determined by this normalizer.

Let $\mathcal{H} < \mathcal{G}$ and $h_j \in \mathcal{H}$. Then $h_j^{-1}\mathcal{H}h_j = \mathcal{H}$ holds because $\mathcal{H}$ is a group. If $\mathcal{H} \lhd \mathcal{G}$, then $g_k^{-1}\mathcal{H}g_k = \mathcal{H}$ for any $g_k \in \mathcal{G}$. If $\mathcal{H}$ is not a normal subgroup of $\mathcal{G}$, there may nevertheless be elements $g_p \in \mathcal{G},\ g_p \notin \mathcal{H}$ for which $g_p^{-1}\mathcal{H}g_p = \mathcal{H}$ holds. We consider the set of all elements $g_p \in \mathcal{G}$ that have this property.

Definition 1.2.4.5.1. The set of all elements $g_p \in \mathcal{G}$ that map the subgroup $\mathcal{H} < \mathcal{G}$ onto itself by conjugation, $\mathcal{H} = g_p^{-1}\mathcal{H}g_p = \mathcal{H}^{g_p}$, forms a group $\mathcal{N}_\mathcal{G}(\mathcal{H})$, called the *normalizer of* $\mathcal{H}$ *in* $\mathcal{G}$, where $\mathcal{H} \trianglelefteq \mathcal{N}_\mathcal{G}(\mathcal{H}) \le \mathcal{G}$. □

Remarks:

(1) The group $\mathcal{H} < \mathcal{G}$ is a normal subgroup of $\mathcal{G}$, $\mathcal{H} \lhd \mathcal{G}$, if and only if $\mathcal{N}_\mathcal{G}(\mathcal{H}) = \mathcal{G}$.
(2) Let $\mathcal{N}_\mathcal{G}(\mathcal{H}) = \{g_p\}$. One can decompose $\mathcal{G}$ into right cosets relative to $\mathcal{N}_\mathcal{G}(\mathcal{H})$. All elements $g_p\, g_r$ of a right coset $\mathcal{N}_\mathcal{G}(\mathcal{H})\, g_r$ of this decomposition $(\mathcal{G} : \mathcal{N}_\mathcal{G}(\mathcal{H}))$ transform $\mathcal{H}$ into the same subgroup $g_r^{-1}\, g_p^{-1}\, \mathcal{H}\, g_p\, g_r = g_r^{-1}\, \mathcal{H}\, g_r < \mathcal{G}$, which is thus conjugate to $\mathcal{H}$ in $\mathcal{G}$ by g_r.
(3) The elements of different cosets of $(\mathcal{G} : \mathcal{N}_\mathcal{G}(\mathcal{H}))$ transform $\mathcal{H}$ into different conjugates of $\mathcal{H}$. The number of cosets of $\mathcal{N}_\mathcal{G}(\mathcal{H})$ is equal to the index $i_N = |\mathcal{G} : \mathcal{N}_\mathcal{G}(\mathcal{H})|$ of $\mathcal{N}_\mathcal{G}(\mathcal{H})$ in $\mathcal{G}$. Therefore, the number $N_\mathcal{H}$ of conjugates in the conjugacy class of $\mathcal{H}$ is equal to the index i_N and is thus determined by the order of $\mathcal{N}_\mathcal{G}(\mathcal{H})$. From $\mathcal{N}_\mathcal{G}(\mathcal{H}) \ge \mathcal{H}$, $i_N = |\mathcal{G} : \mathcal{N}_\mathcal{G}(\mathcal{H})| \le |\mathcal{G} : \mathcal{H}| = i$ follows. This means that the number of conjugates of a subgroup $\mathcal{H} < \mathcal{G}$ cannot exceed the index $i = |\mathcal{G} : \mathcal{H}|$.
(4) If $\mathcal{H} < \mathcal{G}$ is a maximal subgroup of $\mathcal{G}$, then either $\mathcal{N}_\mathcal{G}(\mathcal{H}) = \mathcal{G}$ and $\mathcal{H} \lhd \mathcal{G}$ is a normal subgroup of $\mathcal{G}$ or $\mathcal{N}_\mathcal{G}(\mathcal{H}) = \mathcal{H}$ and the number of conjugates is equal to the index $i = |\mathcal{G} : \mathcal{H}|$.
(5) For the normalizers of the space groups, see the corresponding part of Section 1.2.6.3.

[4] The right-hand side of this equation is the set of all products $g_r = g_{j_p} g_{k_q}$, where g_{j_p} runs through all elements of $\mathcal{K}_j$ and g_{k_q} through all elements of $\mathcal{K}_k$. Each element g_r is taken only once in the set.

references