1.4. Computer checking of the subgroup data

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1.4.1. Introduction

Most of the data in Part 2 of this volume have been checked by a computer-algebra program. This program is built upon the package *Cryst* (Eick *et al.*, 2001), formerly known as *CrystGAP* (Eick *et al.*, 1997), which in turn is an extension of the computer-algebra program *GAP* (The *GAP* Group, 2002). *GAP* is a program system for computations involving general algebraic structures, in particular groups. Since space groups are infinite groups, the generic algorithms for finite groups often cannot be used for space groups. The *Cryst* package provides the special methods and algorithms needed for working with space groups, thereby extending the field of applicability of *GAP*. The algorithms used in *Cryst* have been described by Eick *et al.* (1997).

1.4.2. Basic capabilities of the *Cryst* package

Before we describe in more detail the checks that have been performed, we briefly summarize the capabilities of the *Cryst* package and how space groups are handled by *Cryst*.

In *Cryst*, a space group is represented as a fixed group of augmented matrices $\mathbb{W}_i$, *cf.* Section 1.2.2.4. The group multiplication therefore coincides with matrix multiplication. The space group is defined by an arbitrary (finite) set of generating elements. The space group consists of the set of all augmented matrices that can be obtained as products of the generating matrices. It is not important which generating elements are chosen. Different sets of generators can define the same space group.

The representation of a space group as a group of augmented matrices implicitly requires the choice of an origin and a vector basis of Euclidean space. The *Cryst* package does not require any particular choice of origin or vector basis. In particular, it is not necessary to work with a lattice basis; in fact, any basis can be used. This has the advantage that one can continue to use the basis chosen for the parent group when passing to a subgroup.

Space groups are conjugated by some affine coordinate transformation if they differ only in their settings, *i.e.* by the choice of their origins and/or vector bases. Such an affine coordinate transformation is also represented by an augmented matrix. Given a coordinate transformation $\mathbb{V}$ and a space group $\mathcal{G}$, the transformed space group $\mathcal{G}'$ is simply generated by the transformed generators of $\mathcal{G}$.

If a space group $\mathcal{G}$ is given by a set of generators, *Cryst* first needs to compute the point group $\mathcal{P}$ of $\mathcal{G}$, and the translation subgroup $\mathcal{T}$, represented by a (canonical) basis of the translation lattice (the basis of the translation lattice chosen by *Cryst* is canonical in the sense that the same basis is always chosen for a given $\mathcal{T}$). The point group $\mathcal{P}$ is simply generated by the linear parts of the generators of $\mathcal{G}$. The mapping H, which sends each element of $\mathcal{G}$ to its linear part, is in fact a group homomorphism H: $\mathcal{G} \longrightarrow \mathcal{P}$. The point group $\mathcal{P}$ is finite and each point-group element p can easily be expressed as a product of generators of $\mathcal{P}$, which are images of generators of $\mathcal{G}$ under H. This provides a way to compute, for any $p \in \mathcal{P}$, some representative pre-image g of p under H, *i.e.* some element $g \in \mathcal{G}$ which is mapped onto p by H. Having expressed the element p as a product of generators, one simply replaces the factors in this product by their representative pre-images under H, which are known by construction, to obtain the representative pre-image g of p.

To compute a generating set of the translation subgroup $\mathcal{T} \leq \mathcal{G}$, one first computes a set of defining relations of the point group $\mathcal{P}$ for the given generators. For a finite group this is a standard task in *GAP*. These defining relations are a set of inequivalent ways to express the identity element of $\mathcal{P}$ by the generators of $\mathcal{P}$. Replacing the factors in such a defining relation by their representative pre-images in $\mathcal{G}$ yields a pre-image of the identity of $\mathcal{P}$, *i.e.* an element of $\mathcal{T}$. The translations so obtained generate, together with the pure translation generators of $\mathcal{G}$, the entire translation subgroup $\mathcal{T} \leq \mathcal{G}$.

We now have all the necessary information to test whether a given augmented matrix $\mathbb{A}$ is an element of a space group $\mathcal{G}$. One first tests whether the linear part M of $\mathbb{A}$ is an element of the point group $\mathcal{P}$ of $\mathcal{G}$. If $M \notin \mathcal{P}$, $\mathbb{A}$ is not an element of $\mathcal{G}$. Otherwise, some pre-image $\mathbb{G}$ of M in $\mathcal{G}$ is computed. $\mathbb{A}$ is then an element of $\mathcal{G}$ if and only if $\mathbb{A}$ and $\mathbb{G}$ differ in their translation part by an element of the translation group $\mathcal{T}$. With this membership test, we can determine whether two space groups are equal, or whether one is a subgroup of the other: a space group $\mathcal{H}$ is a subgroup of a space group $\mathcal{G}$ if all the generators of $\mathcal{H}$ are elements of $\mathcal{G}$. Two space groups are equal if either of them is a subgroup of index 1 of the other.

1.4.3. Computing maximal subgroups

The *Cryst* package has built-in facilities for computing the maximal subgroups of a given index for any space group $\mathcal{G}$. More precisely, given a prime number p, *Cryst* can compute conjugacy-class representatives of those maximal subgroups of $\mathcal{G}$ whose index in $\mathcal{G}$ is a power of p. The algorithms used for this task are described in Eick *et al.* (1997). Essentially, one determines the maximal subgroups of the (finite) factor group $\mathcal{G}/\mathcal{T}_p$, where $\mathcal{T}_p$ is the subgroup of those translations of $\mathcal{G}$ which are a p-fold multiple of an element of the full translation group $\mathcal{T}$ of $\mathcal{G}$. After the maximal subgroups are obtained, the translations in $\mathcal{T}_p$ are added back to the subgroups.

From a representative $\mathcal{H}$ of a conjugacy class of subgroups, the list of all subgroups in the same conjugacy class is obtained by repeatedly conjugating $\mathcal{H}$ (and the subgroups obtained from it by conjugation) with generators of $\mathcal{G}$, until no new conjugated subgroups are obtained. This is also the way of determining whether two subgroups are conjugated: one enumerates the groups in the conjugacy class of one of them, and checks whether the other is among them.

The index of a subgroup $\mathcal{H}$ of $\mathcal{G}$ is easily computed as the product of the index of the point group of $\mathcal{H}$ in the point group of $\mathcal{G}$ and the index of the translation group of $\mathcal{H}$ in the translation group of $\mathcal{G}$. For maximal subgroups, only one of these factors is different from 1. For *klassengleiche* subgroups, the two point groups are equal, whereas for *translationengleiche* subgroups the two translation subgroups are equal. Therefore, a maximal subgroup is easily identified either as a *klassengleiche* or a *translationengleiche* subgroup.

references