

10.1. Crystallographic and noncrystallographic point groups

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10.1.1. Introduction and definitions

A *point group** is a group of symmetry operations all of which leave at least one point unmoved. Thus, all operations containing translations are excluded. Point groups can be subdivided into crystallographic and noncrystallographic point groups. A *crystallographic* point group is a point group that maps a point lattice onto itself. Consequently, rotations and rotoinversions are restricted to the well known crystallographic cases 1, 2, 3, 4, 6 and $\bar{1}$, $\bar{2} = m, \bar{3}, \bar{4}, \bar{6}$ (cf. Chapter 1.3); matrices for these symmetry operations are listed in Tables 11.2.2.1 and 11.2.2.2. No such restrictions apply to the *noncrystallographic* point groups.

The numbers of the crystallographic point groups are finite: 2 for one dimension, 10 for two dimensions and 32 for three dimensions. The numbers of noncrystallographic point groups for dimensions $n \geq 2$ are infinite. The two- and three-dimensional crystallographic point groups and their crystal systems are summarized in Tables 10.1.1.1 and 10.1.1.2. They are described in detail in Section 10.1.2. The two one-dimensional point groups are discussed in Section 2.2.17. The noncrystallographic point groups are treated in Section 10.1.4.

Crystallographic point groups occur:

(i) in vector space as symmetries of the external shapes of crystals, *i.e.* of the set of vectors normal to the crystal faces (morphological symmetry);

(ii) in point space as site symmetries of points in lattices or in crystal structures and as symmetries of atomic groups and coordination polyhedra.

General point groups, *i.e.* crystallographic and noncrystallographic point groups, occur as:

(iii) symmetries of (rigid) molecules (molecular symmetry);

(iv) symmetries of physical properties of crystals (*e.g.* tensor symmetries); here noncrystallographic point groups with axes of order infinity are of particular importance, as in the symmetries of spheres or rotation ellipsoids;

(v) approximate symmetries of the local environment of a point in a crystal structure, *i.e.* as *local* site symmetries. Examples are sphere-like atoms or ions in crystals, as well as icosahedral atomic groups. These noncrystallographic symmetries, however, are only approximate, even for the close neighbourhood of a site.

A (geometric) *crystal class* is the set of all crystals having the same point-group symmetry. The word 'class', therefore, denotes a classificatory pigeon-hole and should not be used as synonymous with the point group of a particular crystal. The symbol of a crystal class is that of the common point group. (For geometric and arithmetic crystal classes of space groups, see Sections 8.2.3 and 8.2.4.)

Of particular importance for the structure determination of crystals are the 11 *centrosymmetric crystallographic point groups*, because they describe the possible symmetries of the diffraction record of a crystal: $\bar{1}$; $2/m$; mmm ; $4/m$; $4/mmm$; $\bar{3}$; $\bar{3}m$; $6/m$; $6/mmm$; $m\bar{3}$; $m\bar{3}m$. This is due to Friedel's rule, which states that, provided anomalous dispersion is neglected, the diffraction record of any crystal is centrosymmetric, even if the crystal is noncentrosymmetric. The symmetry of the diffraction record determines the *Laue class* of the crystal; this is further explained

Table 10.1.1.1. The ten two-dimensional crystallographic point groups, arranged according to crystal system

Dashed lines separate point groups with different Laue classes within one crystal system.

General symbol	Crystal system			
	Oblique (top)	Rectangular (bottom)	Square	Hexagonal
n	1	2	4	3 6
mmm	m	$2mm$	$4mm$	$3m$ $6mm$

in Part 3. For a given crystal, its Laue class is obtained if a symmetry centre is added to its point group, as shown in Table 10.2.1.1.

In two dimensions, six 'centrosymmetric' crystallographic point groups and hence six two-dimensional Laue classes exist: 2; $2mm$; 4; $4mm$; 6; $6mm$. These point groups are, for instance, the only possible symmetries of zero-layer X-ray photographs.

Among the centrosymmetric crystallographic point groups in three dimensions, the *lattice point groups* (holohedral point groups, *holohedries*) are of special importance because they constitute the seven possible point symmetries of lattices, *i.e.* the site symmetries of their nodes. In three dimensions, the seven holohedries are: $\bar{1}$; $2/m$; mmm ; $4/mmm$; $\bar{3}m$; $6/mmm$; $m\bar{3}m$. Note that $\bar{3}m$ is the point symmetry of the rhombohedral lattice and $6/mmm$ the point symmetry of the hexagonal lattice; both occur in the hexagonal crystal family (cf. Chapter 2.1). Point groups that are, within a crystal family, subgroups of a holohedry are called *merohedries*; they are called specifically *hemihedries* for subgroups of index 2, *tetartohedries* for index 4 and *ogdohedries* for index 8.

In two dimensions, four holohedries exist: 2; $2mm$; $4mm$; $6mm$. Note that the hexagonal crystal family in two dimensions contains only one lattice type, with point symmetry $6mm$.

Another classification of the crystallographic point groups is that into isomorphism classes. Here all those point groups that have the same kind of group table appear in one class. These isomorphism classes are also known under the name of *abstract point groups*.

There are 18 abstract crystallographic point groups in three dimensions: the point groups in each of the following lines are isomorphous and belong to the same abstract group:

Order 1 : 1	Order 8 : $422, 4mm, \bar{4}2m$
2 : $\bar{1}, 2, m$	12 : $6/m$
3 : 3	12 : $\bar{3}m, 622, 6mm, \bar{6}2m$
4 : $2/m, 222, mm2$	12 : 23
4 : $4, \bar{4}$	16 : $4/mmm$
6 : $\bar{3}, 6, \bar{6}$	24 : $6/mmm$
6 : $32, 3m$	24 : $m\bar{3}$
8 : mmm	24 : $432, \bar{4}3m$
8 : $4/m$	48 : $m\bar{3}m$.

In two dimensions, the ten crystallographic point groups form nine abstract groups; the groups 2 and m are isomorphous and belong to the same abstract group, the remaining eight point groups correspond to one abstract group each.

* For reasons of simplicity, in this chapter the same term 'point group' is used for a 'particular point group' and a 'type of point group'. For space groups, this distinction is explained in Section 8.2.2. For a different use of the term 'point group' see Section 8.1.6.

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.1.2. *The 32 three-dimensional crystallographic point groups, arranged according to crystal system (cf. Chapter 2.1)*

Full Hermann–Mauguin (left) and Schoenflies symbols (right). Dashed lines separate point groups with different Laue classes within one crystal system.

General symbol	Crystal system											
	Triclinic		Monoclinic (top) Orthorhombic (bottom)		Tetragonal		Trigonal		Hexagonal		Cubic	
n	1	C_1	2	C_2	4	C_4	3	C_3	6	C_6	23	T
\bar{n}	$\bar{1}$	C_i	$m \equiv \bar{2}$	C_s	$\bar{4}$	S_4	$\bar{3}$	C_{3i}	$\bar{6} \equiv 3/m$	C_{3h}	–	–
n/m			$2/m$	C_{2h}	$4/m$	C_{4h}	–	–	$6/m$	C_{6h}	$2/m\bar{3}$	T_h
$n22$			222	D_2	422	D_4	32	D_3	622	D_6	432	O
mmm			$mm2$	C_{2v}	$4mm$	C_{4v}	$3m$	C_{3v}	$6mm$	C_{6v}	–	–
$\bar{n}2m$			–	–	$\bar{4}2m$	D_{2d}	$\bar{3}2/m$	D_{3d}	$\bar{6}2m$	D_{3h}	$\bar{4}3m$	T_d
$n/m\ 2/m\ 2/m$			$2/m\ 2/m\ 2/m$	D_{2h}	$4/m\ 2/m\ 2/m$	D_{4h}	–	–	$6/m\ 2/m\ 2/m$	D_{6h}	$4/m\ \bar{3}\ 2/m$	O_h

10.1.2. Crystallographic point groups

10.1.2.1. Description of point groups

In crystallography, point groups usually are described

- (i) by means of their Hermann–Mauguin or Schoenflies symbols;
- (ii) by means of their stereographic projections;
- (iii) by means of the matrix representations of their symmetry operations, frequently listed in the form of Miller indices (hkl) of the equivalent general crystal faces;

(iv) by means of drawings of actual crystals, natural or synthetic. Descriptions (i) through (iii) are given in this section, whereas for crystal drawings and actual photographs reference is made to textbooks of crystallography and mineralogy; this also applies to the construction and the properties of the stereographic projection.

In Tables 10.1.2.1 and 10.1.2.2, the two- and three-dimensional crystallographic point groups are listed and described. The tables are arranged according to crystal systems and Laue classes. Within each crystal system and Laue class, the sequence of the point groups corresponds to that in the space-group tables of this volume: pure rotation groups are followed by groups containing reflections, rotoinversions and inversions. The holohedral point group is always given last.

In Tables 10.1.2.1 and 10.1.2.2, some point groups are described in *two or three versions*, in order to bring out the relations to the corresponding space groups (cf. Section 2.2.3):

(a) The three monoclinic point groups 2 , m and $2/m$ are given with two settings, one with ‘unique axis b ’ and one with ‘unique axis c ’.

(b) The two point groups $\bar{4}2m$ and $\bar{6}m2$ are described for two orientations with respect to the crystal axes, as $\bar{4}2m$ and $\bar{4}m2$ and as $\bar{6}m2$ and $\bar{6}2m$.

(c) The five trigonal point groups 3 , $\bar{3}$, 32 , $3m$ and $\bar{3}m$ are treated with two axial systems, ‘hexagonal axes’ and ‘rhombohedral axes’.

(d) The hexagonal-axes description of the three trigonal point groups 32 , $3m$ and $\bar{3}m$ is given for two orientations, as 321 and 312 , as $3m1$ and $31m$, and as $\bar{3}m1$ and $\bar{3}1m$; this applies also to the two-dimensional point group $3m$.

The presentation of the point groups is similar to that of the space groups in Part 7. The *headline* contains the short Hermann–Mauguin and the Schoenflies symbols. The full Hermann–Mauguin symbol, if different, is given below the short symbol. No Schoenflies symbols exist for two-dimensional groups. For an explanation of the symbols see Section 2.2.4 and Chapter 12.1.

Next to the headline, a pair of *stereographic projections* is given. The diagram on the left displays a general crystal or point form, that on the right shows the ‘framework of symmetry elements’. Except as noted below, the c axis is always normal to the plane of the figure,

the a axis points down the page and the b axis runs horizontally from left to right. For the five trigonal point groups, the c axis is normal to the page only for the description with ‘hexagonal axes’; if described with ‘rhombohedral axes’, the direction $[111]$ is normal and the positive a axis slopes towards the observer. The conventional coordinate systems used for the various crystal systems are listed in Table 2.1.2.1 and illustrated in Figs. 2.2.6.1 to 2.2.6.10.

In the *right-hand projection*, the graphical symbols of the symmetry elements are the same as those used in the space-group diagrams; they are listed in Chapter 1.4. Note that the symbol of a symmetry centre, a small circle, is also used for a face-pole in the left-hand diagram. Mirror planes are indicated by heavy solid lines or circles; thin lines are used for the projection circle, for symmetry axes in the plane and for some special zones in the cubic system.

In the *left-hand projection*, the projection circle and the coordinate axes are indicated by thin solid lines, as are again some special zones in the cubic system. The dots and circles in this projection can be interpreted in two ways.

(i) As *general face poles*, where they represent general crystal faces which form a polyhedron, the ‘general crystal form’ (face form) $\{hkl\}$ of the point group (see below). In two dimensions, edges, edge poles, edge forms and polygons take the place of faces, face poles, crystal forms (face forms) and polyhedra in three dimensions.

Face poles marked as dots lie above the projection plane and represent faces which intersect the positive c axis* (positive Miller index l), those marked as circles lie below the projection plane (negative Miller index l). In two dimensions, edge poles always lie on the pole circle.

(ii) As *general points* (centres of atoms) that span a polyhedron or polygon, the ‘general crystallographic point form’ x, y, z . This interpretation is of interest in the study of coordination polyhedra, atomic groups and molecular shapes. The polyhedron or polygon of a point form is dual to the polyhedron of the corresponding crystal form.†

The general, special and limiting *crystal forms* and *point forms* constitute the main part of the table for each point group. The theoretical background is given below under *Crystal and point forms*; the explanation of the listed data is to be found under *Description of crystal and point forms*.

* This does not apply to ‘rhombohedral axes’: here the positive directions of all three axes slope upwards from the plane of the paper: cf. Fig. 2.2.6.9.

† Dual polyhedra have the same number of edges, but the numbers of faces and vertices are interchanged; cf. textbooks of geometry.

10. POINT GROUPS AND CRYSTAL CLASSES

The last entry for each point group contains the *Symmetry of special projections*, i.e. the plane point group that is obtained if the three-dimensional point group is projected along a symmetry direction. The special projection directions are the same as for the space groups; they are listed in Section 2.2.14. The relations between the axes of the three-dimensional point group and those of its two-dimensional projections can easily be derived with the help of the stereographic projection. No projection symmetries are listed for the two-dimensional point groups.

Note that the symmetry of a projection along a certain direction may be higher than the symmetry of the crystal face normal to that direction. For example, in point group $\bar{1}$ all faces have face symmetry 1, whereas projections along any direction have symmetry 2; in point group 422, the faces (001) and (00 $\bar{1}$) have face symmetry 4, whereas the projection along [001] has symmetry 4mm.

10.1.2.2. Crystal and point forms

For a point group \mathcal{P} a *crystal form* is a set of all symmetrically equivalent faces; a *point form* is a set of all symmetrically equivalent points. Crystal and point forms in point groups correspond to 'crystallographic orbits' in space groups; cf. Section 8.3.2.

Two kinds of crystal and point forms with respect to \mathcal{P} can be distinguished. They are defined as follows:

(i) *General form*: A *face* is called 'general' if only the identity operation transforms the face onto itself. Each complete set of symmetrically equivalent 'general faces' is a *general crystal form*. The *multiplicity* of a general form, i.e. the number of its faces, is the order of \mathcal{P} . In the stereographic projection, the poles of general faces do *not* lie on symmetry elements of \mathcal{P} .

A *point* is called 'general' if its *site symmetry*, i.e. the group of symmetry operations that transforms this point onto itself, is 1. A *general point form* is a complete set of symmetrically equivalent 'general points'.

(ii) *Special form*: A *face* is called 'special' if it is transformed onto itself by at least one symmetry operation of \mathcal{P} , in addition to the identity. Each complete set of symmetrically equivalent 'special faces' is called a *special crystal form*. The *face symmetry* of a special face is the group of symmetry operations that transforms this face onto itself; it is a subgroup of \mathcal{P} . The multiplicity of a special crystal form is the multiplicity of the general form divided by the order of the face-symmetry group. In the stereographic projection, the poles of special faces lie on symmetry elements of \mathcal{P} . The Miller indices of a special crystal form obey restrictions like $\{hk0\}$, $\{hhl\}$, $\{100\}$.

A *point* is called 'special' if its *site symmetry* is higher than 1. A *special point form* is a complete set of symmetrically equivalent 'special points'. The multiplicity of a special point form is the multiplicity of the general form divided by the order of the site-symmetry group. It is thus the same as that of the corresponding special crystal form. The coordinates of the points of a special point form obey restrictions, like $x, y, 0$; x, x, z ; $x, 0, 0$. The point $0, 0, 0$ is not considered to be a point form.

In two dimensions, point groups 1, 2, 3, 4 and 6 and, in three dimensions, point groups 1 and $\bar{1}$ have no special crystal and point forms.

General and special crystal and point forms can be represented by their sets of equivalent Miller indices $\{hkl\}$ and point coordinates x, y, z . Each set of these 'triplets' stands for infinitely many crystal forms or point forms which are obtained by independent variation of the values and signs of the Miller indices h, k, l or the point coordinates x, y, z .

It should be noted that for crystal forms, owing to the well known 'law of rational indices', the indices h, k, l must be integers; no such

restrictions apply to the coordinates x, y, z , which can be rational or irrational numbers.

Example

In point group 4, the general crystal form $\{hkl\}$ stands for the set of all possible tetragonal pyramids, pointing either upwards or downwards, depending on the sign of l ; similarly, the general point form x, y, z includes all possible squares, lying either above or below the origin, depending on the sign of z . For the limiting cases $l = 0$ or $z = 0$, see below.

In order to survey the infinite number of possible forms of a point group, they are classified into *Wyckoff positions of crystal and point forms*, for short *Wyckoff positions*. This name has been chosen in analogy to the Wyckoff positions of space groups; cf. Sections 2.2.11 and 8.3.2. In point groups, the term 'position' can be visualized as the position of the face poles and points in the stereographic projection. Each 'Wyckoff position' is labelled by a *Wyckoff letter*.

Definition

A 'Wyckoff position of crystal and point forms' consists of all those crystal forms (point forms) of a point group \mathcal{P} for which the face poles (points) are positioned on the same set of conjugate symmetry elements of \mathcal{P} ; i.e. for each face (point) of one form there is one face (point) of every other form of the same 'Wyckoff position' that has exactly the same face (site) symmetry.

Each point group contains one 'general Wyckoff position' comprising all *general* crystal and point forms. In addition, up to two 'special Wyckoff positions' may occur in two dimensions and up to six in three dimensions. They are characterized by the different sets of conjugate face and site symmetries and correspond to the seven positions of a pole in the interior, on the three edges, and at the three vertices of the so-called 'characteristic triangle' of the stereographic projection.

Examples

- (1) All tetragonal pyramids $\{hkl\}$ and tetragonal prisms $\{hk0\}$ in point group 4 have face symmetry 1 and belong to the same general 'Wyckoff position' $4b$, with Wyckoff letter b .
- (2) All tetragonal pyramids and tetragonal prisms in point group 4mm belong to two special 'Wyckoff positions', depending on the orientation of their face-symmetry groups m with respect to the crystal axes: For the 'oriented face symmetry' $.m.$, the forms $\{h0l\}$ and $\{100\}$ belong to Wyckoff position $4c$; for the oriented face symmetry $..m$, the forms $\{hhl\}$ and $\{110\}$ belong to Wyckoff position $4b$. The face symmetries $.m.$ and $..m$ are not conjugate in point group 4mm. For the analogous 'oriented site symmetries' in space groups, see Section 2.2.12.

It is instructive to subdivide the crystal forms (point forms) of one Wyckoff position further, into *characteristic* and *noncharacteristic* forms. For this, one has to consider two symmetries that are connected with each crystal (point) form:

- (i) the point group \mathcal{P} by which a form is generated (*generating point group*), i.e. the point group in which it occurs;
- (ii) the full symmetry (inherent symmetry) of a form (considered as a polyhedron by itself), here called *eigensymmetry* \mathcal{C} . The *eigensymmetry point group* \mathcal{C} is either the generating point group itself or a supergroup of it.

Examples

- (1) Each tetragonal pyramid $\{hkl\}$ ($l \neq 0$) of Wyckoff position $4b$ in point group 4 has generating symmetry 4 and *eigensymmetry*

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

$4mm$; each tetragonal prism $\{hk0\}$ of the same Wyckoff position has generating symmetry 4 again, but *eigensymmetry* $4/mmm$.

- (2) A cube $\{100\}$ may have generating symmetry $23, m\bar{3}, 432, \bar{4}3m$ or $m\bar{3}m$, but its *eigensymmetry* is always $m\bar{3}m$.

The *eigensymmetries* and the generating symmetries of the 47 crystal forms (point forms) are listed in Table 10.1.2.3. With the help of this table, one can find the various point groups in which a given crystal form (point form) occurs, as well as the face (site) symmetries that it exhibits in these point groups; for experimental methods see Sections 10.2.2 and 10.2.3.

With the help of the two groups \mathcal{P} and \mathcal{C} , each crystal or point form occurring in a particular point group can be assigned to one of the following two categories:

- (i) *characteristic* form, if *eigensymmetry* \mathcal{C} and generating symmetry \mathcal{P} are the same;
- (ii) *noncharacteristic* form, if \mathcal{C} is a proper supergroup of \mathcal{P} .

The importance of this classification will be apparent from the following examples.

Examples

- (1) A pedion and a pinacoid are noncharacteristic forms in all crystallographic point groups in which they occur;
- (2) all other crystal or point forms occur as characteristic forms in their *eigensymmetry* group \mathcal{C} ;
- (3) a tetragonal pyramid is noncharacteristic in point group 4 and characteristic in $4mm$;
- (4) a hexagonal prism can occur in nine point groups (12 Wyckoff positions) as a noncharacteristic form; in $6/mmm$, it occurs in two Wyckoff positions as a characteristic form.

The general forms of the 13 point groups with no, or only one, symmetry direction ('monoaxial groups') 1, 2, 3, 4, 6, $\bar{1}$, m , $\bar{3}$, 4, $\bar{6} = 3/m, 2/m, 4/m, 6/m$ are always noncharacteristic, *i.e.* their *eigensymmetries* are enhanced in comparison with the generating point groups. The general positions of the other 19 point groups always contain characteristic crystal forms that may be used to determine the point group of a crystal uniquely (*cf.* Section 10.2.2).*

So far, we have considered the occurrence of one crystal or point form in different point groups and different Wyckoff positions. We now turn to the occurrence of different kinds of crystal or point forms in one and the same Wyckoff position of a particular point group.

In a Wyckoff position, crystal forms (point forms) of different *eigensymmetries* may occur; the crystal forms (point forms) with the lowest *eigensymmetry* (which is always well defined) are called *basic forms* (German: *Grundformen*) of that Wyckoff position. The crystal and point forms of higher *eigensymmetry* are called *limiting forms* (German: *Grenzformen*) (*cf.* Table 10.1.2.3). These forms are always noncharacteristic.

Limiting forms[†] occur for certain restricted values of the Miller indices or point coordinates. They always have the same multiplicity and oriented face (site) symmetry as the corresponding basic forms because they belong to the same Wyckoff position. The

enhanced *eigensymmetry* of a limiting form may or may not be accompanied by a change in the topology[‡] of its polyhedra, compared with that of a basic form. In every case, however, the name of a limiting form is different from that of a basic form.

The face poles (or points) of a limiting form lie on symmetry elements of a supergroup of the point group that are not symmetry elements of the point group itself. There may be several such supergroups.

Examples

- (1) In point group 4, the (noncharacteristic) crystal forms $\{hkl\}$ ($l \neq 0$) (tetragonal pyramids) of *eigensymmetry* $4mm$ are basic forms of the general Wyckoff position $4b$, whereas the forms $\{hk0\}$ (tetragonal prisms) of higher *eigensymmetry* $4/mmm$ are 'limiting general forms'. The face poles of forms $\{hk0\}$ lie on the horizontal mirror plane of the supergroup $4/m$.
- (2) In point group $4mm$, the (characteristic) special crystal forms $\{h0l\}$ with *eigensymmetry* $4mm$ are 'basic forms' of the special Wyckoff position $4c$, whereas $\{100\}$ with *eigensymmetry* $4/mmm$ is a 'limiting special form'. The face poles of $\{100\}$ are located on the intersections of the vertical mirror planes of the point group $4mm$ with the horizontal mirror plane of the supergroup $4/mmm$, *i.e.* on twofold axes of $4/mmm$.

Whereas basic and limiting forms belonging to one 'Wyckoff position' are always clearly distinguished, closer inspection shows that a Wyckoff position may contain different 'types' of limiting forms. We need, therefore, a further criterion to classify the limiting forms of one Wyckoff position into types: A *type of limiting form of a Wyckoff position* consists of all those limiting forms for which the face poles (points) are located on the same set of additional conjugate symmetry elements of the holohedral point group (for the trigonal point groups, the hexagonal holohedry $6/mmm$ has to be taken). Different types of limiting forms may have the same *eigensymmetry* and the same topology, as shown by the examples below. The occurrence of two topologically different polyhedra as two 'realizations' of one type of limiting form in point groups $23, m\bar{3}$ and 432 is explained below in Section 10.1.2.4, *Notes on crystal and point forms*, item (viii).

Examples

- (1) In point group 32, the limiting general crystal forms are of four types:
 - (i) ditrigonal prisms, *eigensymmetry* $\bar{6}2m$ (face poles on horizontal mirror plane of holohedry $6/mmm$);
 - (ii) trigonal dipyrramids, *eigensymmetry* $62m$ (face poles on one kind of vertical mirror plane);
 - (iii) rhombohedra, *eigensymmetry* $\bar{3}m$ (face poles on second kind of vertical mirror plane);
 - (iv) hexagonal prisms, *eigensymmetry* $6/mmm$ (face poles on horizontal twofold axes).

Types (i) and (ii) have the same *eigensymmetry* but different topologies; types (i) and (iv) have the same topology but different *eigensymmetries*; type (iii) differs from the other three types in both *eigensymmetry* and topology.

- (2) In point group 222, the face poles of the three types of general limiting forms, rhombic prisms, are located on the three (non-equivalent) symmetry planes of the holohedry mmm . Geometrically, the axes of the prisms are directed along the three non-equivalent orthorhombic symmetry directions. The three types

* For a survey of these relations, as well as of the 'limiting forms', it is helpful to consider the (seven) *normalizers* of the crystallographic point groups in the group of all rotations and reflections (orthogonal group, sphere group); normalizers of the crystallographic and noncrystallographic point groups are listed in Tables 15.4.1.1 and 15.4.1.2.

† The treatment of 'limiting forms' in the literature is quite ambiguous. In some textbooks, limiting forms are omitted or treated as special forms in their own right; other authors define only limiting *general* forms and consider limiting *special* forms as if they were new special forms. For additional reading, see P. Niggli (1941, pp. 80–98).

‡ The topology of a polyhedron is determined by the numbers of its vertices, edges and faces, by the number of vertices of each face and by the number of faces meeting in each vertex.

10. POINT GROUPS AND CRYSTAL CLASSES

of limiting forms have the same *eigensymmetry* and the same topology but different orientations.

Similar cases occur in point groups 422 and 622 (*cf.* Table 10.1.2.3, footnote *).

Not considered in this volume are limiting forms of another kind, namely those that require either special metrical conditions for the axial ratios or irrational indices or coordinates (which always can be closely approximated by rational values). For instance, a rhombic disphenoid can, for special axial ratios, appear as a tetragonal or even as a cubic tetrahedron; similarly, a rhombohedron can degenerate to a cube. For special irrational indices, a ditetragonal prism changes to a (noncrystallographic) octagonal prism, a dihexagonal pyramid to a dodecagonal pyramid or a crystallographic pentagon-dodecahedron to a regular pentagon-dodecahedron. These kinds of limiting forms are listed by A. Niggli (1963).

In conclusion, each general or special Wyckoff position always contains one set of basic crystal (point) forms. In addition, it may contain one or more sets of limiting forms of different types. As a rule,† each type comprises polyhedra of the same *eigensymmetry* and topology and, hence, of the same name, for instance ‘ditetragonal pyramid’. The name of the *basic general* forms is often used to designate the corresponding crystal class, for instance ‘ditetragonal-pyramidal class’; some of these names are listed in Table 10.1.2.4.

10.1.2.3. Description of crystal and point forms

The main part of each point-group table describes the general and special *crystal and point forms* of that point group, in a manner analogous to the *positions* in a space group. The general Wyckoff position is given at the top, followed downwards by the special Wyckoff positions with decreasing multiplicity. Within each Wyckoff position, the first block refers to the basic forms, the subsequent blocks list the various types of limiting form, if any.

The columns, from left to right, contain the following data (further details are to be found below in Section 10.1.2.4, *Notes on crystal and point forms*):

Column 1: *Multiplicity* of the ‘Wyckoff position’, *i.e.* the number of equivalent faces and points of a crystal or point form.

Column 2: *Wyckoff letter*. Each general or special ‘Wyckoff position’ is designated by a ‘Wyckoff letter’, analogous to the Wyckoff letter of a position in a space group (*cf.* Section 2.2.11).

Column 3: *Face symmetry* or *site symmetry*, given in the form of an ‘oriented point-group symbol’, analogous to the oriented site-symmetry symbols of space groups (*cf.* Section 2.2.12). The face symmetry is also the symmetry of etch pits, striations and other face markings. For the two-dimensional point groups, this column contains the *edge symmetry*, which can be either 1 or *m*.

Column 4: *Name of crystal form*. If more than one name is in common use, several are listed. The names of the limiting forms are also given. The crystal forms, their names, *eigensymmetries* and occurrence in the point groups are summarized in Table 10.1.2.3, which may be useful for determinative purposes, as explained in Sections 10.2.2 and 10.2.3. There are 47 different types of crystal form. Frequently, 48 are quoted because ‘sphenoid’ and ‘dome’ are considered as two different forms. It is customary, however, to regard them as the same form, with the name ‘dihedron’.

Name of point form (printed in italics). There exists no general convention on the names of the point forms. Here, only one name is given, which does not always agree with that of other authors. The names of the point forms are also contained in Table 10.1.2.3. Note

† For the exceptions in the cubic crystal system *cf.* Section 10.1.2.4, *Notes on crystal and point forms*, item (viii)

that the same point form, ‘line segment’, corresponds to both sphenoid and dome. The letter in parentheses after each name of a point form is explained below.

Column 5: *Miller indices* (*hkl*) for the symmetrically equivalent faces (edges) of a crystal form. In the trigonal and hexagonal crystal systems, when referring to hexagonal axes, Bravais–Miller indices (*hkil*) are used, with $h + k + i = 0$.

Coordinates *x, y, z* for the symmetrically equivalent points of a point form are not listed explicitly because they can be obtained from data in this volume as follows: after the name of the point form, a letter is given in parentheses. This is the Wyckoff letter of the corresponding position in the symmorphic *P* space group that belongs to the point group under consideration. The coordinate triplets of this (general or special) position apply to the point form of the point group.

The triplets of Miller indices (*hkl*) and point coordinates *x, y, z* are arranged in such a way as to show analogous sequences; they are both based on the same set of generators, as described in Sections 2.2.10 and 8.3.5. For all point groups, except those referred to a hexagonal coordinate system, the correspondence between the (*hkl*) and the *x, y, z* triplets is immediately obvious.‡

The sets of symmetrically equivalent crystal faces also represent the sets of equivalent reciprocal-lattice points, as well as the sets of equivalent X-ray (neutron) reflections.

Examples

- (1) In point group $\bar{4}$, the general crystal form 4*b* is listed as (*hkl*) ($\bar{h}kl$) (*kh* \bar{l}) ($\bar{k}h\bar{l}$): the corresponding general position 4*h* of the symmorphic space group $P\bar{4}$ reads *x, y, z; x, $\bar{y}, z; y, \bar{x}, z; \bar{y}, x, z$.*
- (2) In point group 3, the general crystal form 3*b* is listed as (*hkl*) (*ihkl*) (*kihl*) with $i = -(h + k)$; the corresponding general position 3*d* of the symmorphic space group $P3$ reads *x, y, z; $\bar{y}, x - y, z; -x + y, \bar{x}, z$.*
- (3) The Miller indices of the *cubic point groups* are arranged in one, two or four blocks of (3 × 4) entries. The first block (upper left) belongs to point group 23. The second block (upper right) belongs to the diagonal twofold axes in 432 and $m\bar{3}m$ or to the diagonal mirror plane in $\bar{4}3m$. In point groups $m\bar{3}$ and $m\bar{3}m$, the lower one or two blocks are derived from the upper blocks by application of the inversion.

10.1.2.4. Notes on crystal and point forms

(i) As mentioned in Section 10.1.2.2, each set of Miller indices of a given point group represents infinitely many face forms with the same name. Exceptions occur for the following cases.

Some special crystal forms occur with only *one* representative. Examples are the pinacoid {001}, the hexagonal prism {10 $\bar{1}$ 0} and the cube {100}. The Miller indices of these forms consist of fixed numbers and signs and contain no variables.

In a few noncentrosymmetric point groups, a special crystal form is realized by *two* representatives: they are related by a centre of symmetry that is not part of the point-group symmetry. These cases are

- (a) the two pedions (001) and (00 $\bar{1}$);

‡ The matrices of corresponding triplets ($\bar{h}\bar{k}\bar{l}$) and $\bar{x}, \bar{y}, \bar{z}$, *i.e.* of triplets generated by the same symmetry operation from (*hkl*) and *x, y, z*, are inverse to each other, provided the *x, y, z* and $\bar{x}, \bar{y}, \bar{z}$ are regarded as columns and the (*hkl*) and ($\bar{h}\bar{k}\bar{l}$) as rows: this is due to the contravariant and covariant nature of the point coordinates and Miller indices, respectively. Note that for orthogonal matrices the inverse matrix equals the transposed matrix; in crystallography, this applies to all coordinate systems (including the rhombohedral one), except for the hexagonal system. The matrices for the symmetry operations occurring in the crystallographic point groups are listed in Tables 11.2.2.1 and 11.2.2.2.

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

(b) the two trigonal prisms $\{10\bar{1}0\}$ and $\{\bar{1}010\}$; similarly for two dimensions;

(c) the two trigonal prisms $\{11\bar{2}0\}$ and $\{\bar{1}\bar{1}20\}$; similarly for two dimensions;

(d) the positive and negative tetrahedra $\{111\}$ and $\{\bar{1}\bar{1}\bar{1}\}$.
In the point-group tables, both representatives of these forms are listed, separated by 'or', for instance '(001) or (00 $\bar{1}$)'.

(ii) In crystallography, the terms tetragonal, trigonal, hexagonal, as well as tetragon, trigon and hexagon, imply that the cross sections of the corresponding polyhedra, or the polygons, are *regular* tetragons (squares), trigons or hexagons. Similarly, ditetragonal, ditrigonal, dihexagonal, as well as ditetragon, ditrison and dihexagon, refer to *semi-regular* cross sections or polygons.

(iii) Crystal forms can be 'open' or 'closed'. A crystal form is 'closed' if its faces form a closed polyhedron; the minimum number of faces for a closed form is 4. Closed forms are disphenoids, dipyramids, rhombohedra, trapezohedra, scalenohedra and all cubic forms; open forms are pedions, pinacoids, sphenoids (domes), pyramids and prisms.

A point form is always closed. It should be noted, however, that a point form dual to a *closed* face form is a *three*-dimensional polyhedron, whereas the dual of an *open* face form is a *two*- or *one*-dimensional polygon, which, in general, is located 'off the origin' but may be centred at the origin (here called 'through the origin').

(iv) Crystal forms are well known; they are described and illustrated in many textbooks. Crystal forms are 'isohedral' polyhedra that have all faces equivalent but may have more than one kind of vertex; they include regular polyhedra. The in-sphere of the polyhedron touches all the faces.

Crystallographic point forms are less known; they are described in a few places only, notably by A. Niggli (1963), by Fischer *et al.* (1973), and by Burzlaff & Zimmermann (1977). The latter publication contains drawings of the polyhedra of all point forms. Point forms are 'isogonal' polyhedra (polygons) that have all vertices equivalent but may have more than one kind of face;* again, they include regular polyhedra. The circumsphere of the polyhedron passes through all the vertices.

In most cases, the names of the point-form polyhedra can easily be derived from the corresponding crystal forms: the duals of n -gonal pyramids are regular n -gons off the origin, those of n -gonal prisms are regular n -gons through the origin. The duals of di- n -gonal pyramids and prisms are truncated (regular) n -gons, whereas the duals of n -gonal dipyramids are n -gonal prisms.

In a few cases, however, the relations are not so evident. This applies mainly to some cubic point forms [see item (v) below]. A further example is the rhombohedron, whose dual is a trigonal antiprism (in general, the duals of n -gonal streptohedra are n -gonal antiprisms).† The duals of n -gonal trapezohedra are polyhedra intermediate between n -gonal prisms and n -gonal antiprisms; they are called here 'twisted n -gonal antiprisms'. Finally, the duals of di- n -gonal scalenohedra are n -gonal antiprisms 'sliced off' perpendicular to the prism axis by the pinacoid $\{001\}$.‡

(v) Some cubic point forms have to be described by 'combinations' of 'isohedral' polyhedra because no common

names exist for 'isogonal' polyhedra. The maximal number of polyhedra required is three. The *shape* of the combination that describes the point form depends on the relative sizes of the polyhedra involved, *i.e.* on the relative values of their central distances. Moreover, in some cases even the *topology* of the point form may change.

Example

'Cube truncated by octahedron' and 'octahedron truncated by cube'. Both forms have 24 vertices, 14 faces and 36 edges but the faces of the first combination are octagons and trigons, those of the second are hexagons and tetragons. These combinations represent different special point forms x, x, z and $0, y, z$. One form can change into the other only *via* the (semi-regular) cuboctahedron $0, y, y$, which has 12 vertices, 14 faces and 24 edges.

The unambiguous description of the cubic point forms by combinations of 'isohedral' polyhedra requires restrictions on the relative sizes of the polyhedra of a combination. The permissible range of the size ratios is limited on the one hand by vanishing, on the other hand by splitting of vertices of the combination. Three cases have to be distinguished:

(a) The relative sizes of the polyhedra of the combination can vary *independently*. This occurs whenever three edges meet in one vertex. In Table 10.1.2.2, the names of these point forms contain the term 'truncated'.

Examples

- (1) 'Octahedron truncated by cube' (24 vertices, dual to tetrahexahedron).
- (2) 'Cube truncated by two tetrahedra' (24 vertices, dual to hexatetrahedron), implying independent variation of the relative sizes of the two truncating tetrahedra.

(b) The relative sizes of the polyhedra are *interdependent*. This occurs for combinations of three polyhedra whenever four edges meet in one vertex. The names of these point forms contain the symbol '&'.

Example

'Cube & two tetrahedra' (12 vertices, dual to tetragon-tritetrahedron); here the interdependence results from the requirement that in the combination a cube edge is reduced to a vertex in which faces of the two tetrahedra meet. The location of this vertex on the cube edge is free. A higher symmetrical 'limiting' case of this combination is the 'cuboctahedron', where the two tetrahedra have the same sizes and thus form an octahedron.

(c) The relative sizes of the polyhedra are *fixed*. This occurs for combinations of three polyhedra if five edges meet in one vertex. These point forms are designated by special names (snub tetrahedron, snub cube, irregular icosahedron), or their names contain the symbol '+'.
The cuboctahedron appears here too, as a limiting form of the snub tetrahedron (dual to pentagon-tritetrahedron) and of the irregular icosahedron (dual to pentagon-dodecahedron) for the special coordinates $0, y, y$.

(vi) Limiting crystal forms result from general or special crystal forms for special values of certain geometrical parameters of the form.

Examples

- (1) A pyramid degenerates into a prism if its apex angle becomes 0, *i.e.* if the apex moves towards infinity.

(continued on page 795)

* Thus, the name 'prism' for a *point form* implies combination of the prism with a pinacoid.

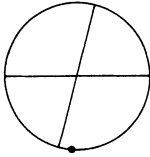
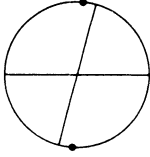
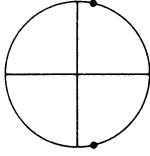
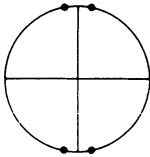
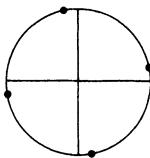
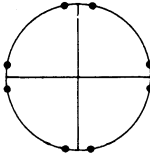
† A tetragonal tetrahedron is a digonal streptohedron; hence, its dual is a 'digonal antiprism', which is again a tetragonal tetrahedron.

‡ The dual of a tetragonal (= di-digonal) scalenohedron is a 'digonal antiprism', which is 'cut off' by the pinacoid $\{001\}$.

10. POINT GROUPS AND CRYSTAL CLASSES

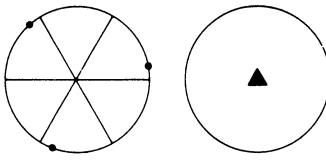
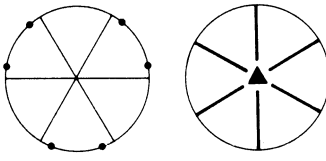
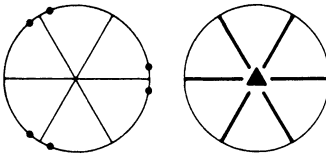
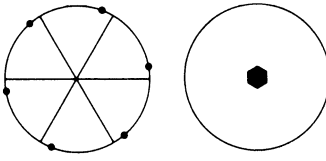
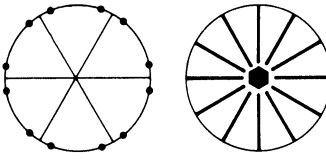
Table 10.1.2.1. *The ten two-dimensional crystallographic point groups*

General, special and limiting edge forms and *point forms* (italics), oriented edge and site symmetries, and Miller indices (*hk*) of equivalent edges [for hexagonal groups Bravais–Miller indices (*hki*) are used if referred to hexagonal axes]; for point coordinates see text.

OBLIQUE SYSTEM				
1				
1	<i>a</i>	1	Single edge <i>Single point (a)</i>	(<i>hk</i>)
2				
2	<i>a</i>	1	Pair of parallel edges <i>Line segment through origin (e)</i>	(<i>hk</i>) ($\bar{h}\bar{k}$)
RECTANGULAR SYSTEM				
<i>m</i>				
2	<i>b</i>	1	Pair of edges <i>Line segment (c)</i>	(<i>hk</i>) ($\bar{h}\bar{k}$)
			Pair of parallel edges <i>Line segment through origin</i>	(10) ($\bar{1}0$)
1	<i>a</i>	<i>.m.</i>	Single edge <i>Single point (a)</i>	(01) or ($0\bar{1}$)
<i>2mm</i>				
4	<i>c</i>	1	Rhomb <i>Rectangle (i)</i>	(<i>hk</i>) ($\bar{h}\bar{k}$) ($\bar{h}k$) (<i>hk</i>)
2	<i>b</i>	<i>.m.</i>	Pair of parallel edges <i>Line segment through origin (g)</i>	(01) ($0\bar{1}$)
2	<i>a</i>	<i>.m.</i>	Pair of parallel edges <i>Line segment through origin (e)</i>	(10) ($\bar{1}0$)
SQUARE SYSTEM				
4				
4	<i>a</i>	1	Square <i>Square (d)</i>	(<i>hk</i>) ($\bar{h}\bar{k}$) ($\bar{k}h$) (<i>kh</i>)
<i>4mm</i>				
8	<i>c</i>	1	Ditetragon <i>Truncated square (g)</i>	(<i>hk</i>) ($\bar{h}\bar{k}$) ($\bar{k}h$) (<i>kh</i>) ($\bar{h}k$) (<i>hk</i>) (<i>kh</i>) ($\bar{k}h$)
4	<i>b</i>	<i>.m.</i>	Square <i>Square (f)</i>	(11) ($\bar{1}\bar{1}$) ($\bar{1}1$) (11)
4	<i>a</i>	<i>.m.</i>	Square <i>Square (d)</i>	(10) ($\bar{1}0$) (01) ($0\bar{1}$)

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

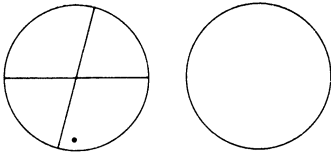
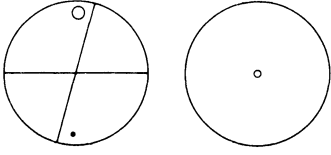
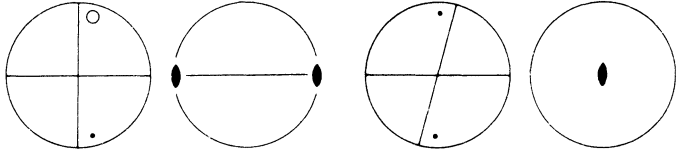
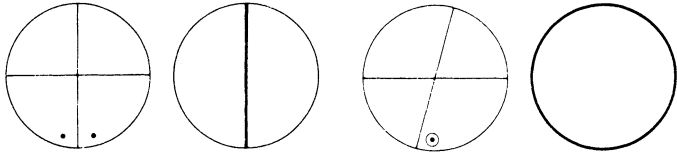
Table 10.1.2.1. *The ten two-dimensional crystallographic point groups (cont.)*

HEXAGONAL SYSTEM				
3				
3	<i>a</i>	1	Trigon Trigon (<i>d</i>)	(<i>hki</i>) (<i>ihk</i>) (<i>kih</i>)
3m1				
6	<i>b</i>	1	Ditrigon Truncated trigon (<i>e</i>)	(<i>hki</i>) (<i>ihk</i>) (<i>kih</i>) (<i>khī</i>) (<i>īkh</i>) (<i>hīk</i>)
			Hexagon Hexagon	(<i>112̄</i>) (<i>211</i>) (<i>12̄1</i>) (<i>112</i>) (<i>211̄</i>) (<i>121̄</i>)
3	<i>a</i>	<i>.m.</i>	Trigon Trigon (<i>d</i>)	(<i>101̄</i>) (<i>110̄</i>) (<i>011̄</i>) or (<i>101</i>) (<i>110</i>) (<i>011</i>)
31m				
6	<i>b</i>	1	Ditrigon Truncated trigon (<i>d</i>)	(<i>hki</i>) (<i>ihk</i>) (<i>kih</i>) (<i>khi</i>) (<i>ikh</i>) (<i>hik</i>)
			Hexagon Hexagon	(<i>101̄</i>) (<i>110̄</i>) (<i>011̄</i>) (<i>011̄</i>) (<i>101</i>) (<i>110</i>)
3	<i>a</i>	<i>.m.</i>	Trigon Trigon (<i>c</i>)	(<i>112̄</i>) (<i>211</i>) (<i>12̄1</i>) or (<i>112</i>) (<i>211̄</i>) (<i>121̄</i>)
6				
6	<i>a</i>	1	Hexagon Hexagon (<i>d</i>)	(<i>hki</i>) (<i>ihk</i>) (<i>kih</i>) (<i>khī</i>) (<i>īkh</i>) (<i>hīk</i>)
6mm				
12	<i>c</i>	1	Dihexagon Truncated hexagon (<i>f</i>)	(<i>hki</i>) (<i>ihk</i>) (<i>kih</i>) (<i>khī</i>) (<i>īkh</i>) (<i>hīk</i>) (<i>khi</i>) (<i>ikh</i>) (<i>hik</i>)
6	<i>b</i>	<i>.m.</i>	Hexagon Hexagon (<i>e</i>)	(<i>101̄</i>) (<i>110̄</i>) (<i>011̄</i>) (<i>101</i>) (<i>110</i>) (<i>011</i>)
6	<i>a</i>	<i>.m.</i>	Hexagon Hexagon (<i>d</i>)	(<i>112̄</i>) (<i>211</i>) (<i>12̄1</i>) (<i>112</i>) (<i>211̄</i>) (<i>121̄</i>)

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups*

General, special and limiting face forms and *point forms* (italics), oriented face and site symmetries, and Miller indices (*hkl*) of equivalent faces [for trigonal and hexagonal groups Bravais–Miller indices (*hkil*) are used if referred to hexagonal axes]; for point coordinates see text.

TRICLINIC SYSTEM			
1	C_1		
1	<i>a</i>	1	Pedion or monohedron <i>Single point (a)</i> Symmetry of special projections Along any direction 1
$\bar{1}$	C_i		
2	<i>a</i>	1	Pinacoid or parallelohedron <i>Line segment through origin (i)</i> Symmetry of special projections Along any direction 2
MONOCLINIC SYSTEM			
2	C_2		
2	<i>b</i>	1	Sphenoid or dihedron <i>Line segment (e)</i> Pinacoid or parallelohedron <i>Line segment through origin</i>
1	<i>a</i>	2	Pedion or monohedron <i>Single point (a)</i> Symmetry of special projections
		Along [100]	Along [010]
Unique axis <i>b</i>	<i>m</i>	2	<i>m</i>
<i>c</i>	<i>m</i>	<i>m</i>	2
		Along [001]	
		<i>m</i>	2
m	C_s		
2	<i>b</i>	1	Dome or dihedron <i>Line segment (c)</i> Pinacoid or parallelohedron <i>Line segment through origin</i>
1	<i>a</i>	<i>m</i>	Pedion or monohedron <i>Single point (a)</i> Symmetry of special projections
		Along [100]	Along [010]
Unique axis <i>b</i>	<i>m</i>	1	<i>m</i>
<i>c</i>	<i>m</i>	<i>m</i>	1
		Along [001]	
		<i>m</i>	1

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

MONOCLINIC SYSTEM (<i>cont.</i>)				
$2/m$	C_{2h}			
				Unique axis b Unique axis c
4	c	1	Rhombic prism <i>Rectangle through origin (o)</i>	(hkl) $(\bar{h}k\bar{l})$ $(\bar{h}k\bar{l})$ (hkl)
2	b	m	Pinacoid or parallelohedron <i>Line segment through origin (m)</i>	$(h0l)$ $(\bar{h}0\bar{l})$
2	a	2	Pinacoid or parallelohedron <i>Line segment through origin (i)</i>	(010) $(0\bar{1}0)$
Symmetry of special projections				
		Along [100]	Along [010]	Along [001]
Unique axis b		$2mm$	2	$2mm$
c		$2mm$	$2mm$	2
ORTHORHOMBIC SYSTEM				
222	D_2			
4	d	1	Rhombic disphenoid or rhombic tetrahedron <i>Rhombic tetrahedron (u)</i>	(hkl) $(\bar{h}k\bar{l})$ $(\bar{h}k\bar{l})$ (hkl)
			Rhombic prism <i>Rectangle through origin</i>	$(hk0)$ $(\bar{h}k0)$ $(\bar{h}k0)$ $(hk0)$
			Rhombic prism <i>Rectangle through origin</i>	$(h0l)$ $(\bar{h}0l)$ $(\bar{h}0l)$ $(h0l)$
			Rhombic prism <i>Rectangle through origin</i>	$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$
2	c	$..2$	Pinacoid or parallelohedron <i>Line segment through origin (q)</i>	(001) $(00\bar{1})$
2	b	$..2$	Pinacoid or parallelohedron <i>Line segment through origin (m)</i>	(010) $(0\bar{1}0)$
2	a	$2..$	Pinacoid or parallelohedron <i>Line segment through origin (i)</i>	(100) $(\bar{1}00)$
Symmetry of special projections				
		Along [100]	Along [010]	Along [001]
		$2mm$	$2mm$	$2mm$

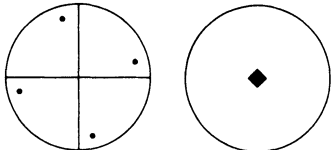
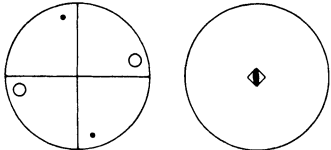
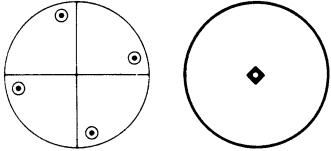
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

ORTHORHOMBIC SYSTEM (cont.)						
$mm2$	C_{2v}					
4	d	1	Rhombic pyramid <i>Rectangle (i)</i>		(hkl) $(\bar{h}\bar{k}l)$ $(h\bar{k}l)$ $(\bar{h}kl)$	
			Rhombic prism <i>Rectangle through origin</i>		$(hk0)$ $(\bar{h}\bar{k}0)$ $(h\bar{k}0)$ $(\bar{h}k0)$	
2	c	$m..$	Dome or dihedron <i>Line segment (g)</i>		$(0kl)$ $(0\bar{k}l)$	
			Pinacoid or parallelohedron <i>Line segment through origin</i>		(010) $(0\bar{1}0)$	
2	b	$.m.$	Dome or dihedron <i>Line segment (e)</i>		$(h0l)$ $(\bar{h}0l)$	
			Pinacoid or parallelohedron <i>Line segment through origin</i>		(100) $(\bar{1}00)$	
1	a	$mm2$	Pedion or monohedron <i>Single point (a)</i>		(001) or $(00\bar{1})$	
			Symmetry of special projections			
			Along $[100]$	Along $[010]$	Along $[001]$	
			m	m	$2mm$	
$m m m$	D_{2h}					
$\frac{2}{m} \frac{2}{m} \frac{2}{m}$						
8	g	1	Rhombic dipyramid <i>Quad (α)</i>		(hkl) $(\bar{h}\bar{k}l)$ $(h\bar{k}\bar{l})$ $(\bar{h}k\bar{l})$ $(\bar{h}kl)$ $(h\bar{k}l)$ $(h\bar{k}\bar{l})$ $(\bar{h}kl)$	
4	f	$..m$	Rhombic prism <i>Rectangle through origin (y)</i>		$(hk0)$ $(\bar{h}\bar{k}0)$ $(h\bar{k}0)$ $(\bar{h}k0)$	
4	e	$.m.$	Rhombic prism <i>Rectangle through origin (w)</i>		$(h0l)$ $(\bar{h}0l)$ $(h0\bar{l})$ $(\bar{h}0\bar{l})$	
4	d	$m..$	Rhombic prism <i>Rectangle through origin (u)</i>		$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$	
2	c	$mm2$	Pinacoid or parallelohedron <i>Line segment through origin (q)</i>		(001) $(00\bar{1})$	
2	b	$m2m$	Pinacoid or parallelohedron <i>Line segment through origin (m)</i>		(010) $(0\bar{1}0)$	
2	a	$2mm$	Pinacoid or parallelohedron <i>Line segment through origin (i)</i>		(100) $(\bar{1}00)$	
			Symmetry of special projections			
			Along $[100]$	Along $[010]$	Along $[001]$	
			$2mm$	$2mm$	$2mm$	

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TETRAGONAL SYSTEM				
4	C_4			
4	b	1	Tetragonal pyramid <i>Square (d)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$
			Tetragonal prism <i>Square through origin</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$
1	a	4..	Pedion or monohedron <i>Single point (a)</i>	(001) or $(00\bar{1})$
			Symmetry of special projections	
		Along $[001]$	Along $[100]$	Along $[110]$
		4	m	m
$\bar{4}$	S_4			
4	b	1	Tetragonal disphenoid or tetragonal tetrahedron <i>Tetragonal tetrahedron (h)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$
			Tetragonal prism <i>Square through origin</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$
2	a	2..	Pinacoid or parallelohedron <i>Line segment through origin (e)</i>	(001) $(00\bar{1})$
			Symmetry of special projections	
		Along $[001]$	Along $[100]$	Along $[110]$
		4	m	m
$4/m$	C_{4h}			
8	c	1	Tetragonal dipyramid <i>Tetragonal prism (l)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$ $(h\bar{k}l)$ $(hk\bar{l})$ (khl) $(\bar{k}\bar{h}l)$
4	b	$m..$	Tetragonal prism <i>Square through origin (j)</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$
2	a	4..	Pinacoid or parallelohedron <i>Line segment through origin (g)</i>	(001) $(00\bar{1})$
			Symmetry of special projections	
		Along $[001]$	Along $[100]$	Along $[110]$
		4	$2mm$	$2mm$

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TETRAGONAL SYSTEM (<i>cont.</i>)					
422	D_4				
8	d	1	Tetragonal trapezohedron <i>Twisted tetragonal antiprism (p)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$ $(\bar{h}\bar{k}l)$ (hkl) (khl) $(\bar{k}\bar{h}l)$	
			Ditetragonal prism <i>Truncated square through origin</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$ $(\bar{h}\bar{k}0)$ (hkl) (khl) $(\bar{k}\bar{h}0)$	
			Tetragonal dipyramid <i>Tetragonal prism</i>	$(h0l)$ $(\bar{h}0l)$ $(0hl)$ $(0\bar{h}l)$ $(\bar{h}0l)$ $(h0l)$ $(0hl)$ $(0\bar{h}l)$	
			Tetragonal dipyramid <i>Tetragonal prism</i>	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}hl)$ $(h\bar{h}l)$ $(\bar{h}\bar{h}l)$ (hhl) (hhl) $(\bar{h}\bar{h}l)$	
4	c	.2.	Tetragonal prism <i>Square through origin (l)</i>	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$	
4	b	..2	Tetragonal prism <i>Square through origin (j)</i>	(110) $(\bar{1}\bar{1}0)$ $(\bar{1}10)$ $(1\bar{1}0)$	
2	a	4..	Pinacoid or parallelohedron <i>Line segment through origin (g)</i>	(001) $(00\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[110]$		
			$4mm$ $2mm$ $2mm$		
$4mm$	C_{4v}				
8	d	1	Ditetragonal pyramid <i>Truncated square (g)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$ $(\bar{h}\bar{k}l)$ (hkl) $(\bar{k}hl)$ (khl)	
			Ditetragonal prism <i>Truncated square through origin</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$ $(\bar{h}\bar{k}0)$ (hkl) (khl) $(\bar{k}\bar{h}0)$	
4	c	. m .	Tetragonal pyramid <i>Square (e)</i>	$(h0l)$ $(\bar{h}0l)$ $(0hl)$ $(0\bar{h}l)$	
			Tetragonal prism <i>Square through origin</i>	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$	
4	b	.. m	Tetragonal pyramid <i>Square (d)</i>	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}hl)$ $(h\bar{h}l)$	
			Tetragonal prism <i>Square through origin</i>	(110) $(\bar{1}\bar{1}0)$ $(\bar{1}10)$ $(1\bar{1}0)$	
1	a	$4mm$	Pedion or monohedron <i>Single point (a)</i>	(001) or $(00\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[110]$		
			$4mm$ m m		

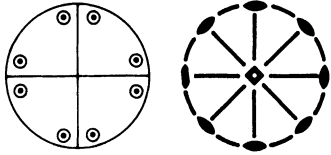
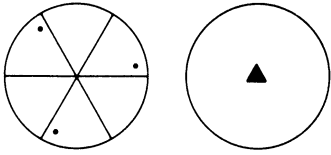
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

 Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TETRAGONAL SYSTEM (<i>cont.</i>)				
$\bar{4}2m$	D_{2d}			
8	<i>d</i>	1	Tetragonal scalenohedron <i>Tetragonal tetrahedron cut off by pinacoid (o)</i> Ditetragonal prism <i>Truncated square through origin</i> Tetragonal dipyramid <i>Tetragonal prism</i>	(hkl) $(\bar{h}\bar{k}l)$ $(k\bar{h}l)$ $(\bar{k}hl)$ $(h\bar{k}l)$ (hkl) $(\bar{k}hl)$ (khl) $(hk0)$ $(\bar{h}\bar{k}0)$ $(k\bar{h}0)$ $(\bar{k}h0)$ $(h\bar{k}0)$ $(hk0)$ $(\bar{k}h0)$ $(kh0)$ $(h0l)$ $(\bar{h}0l)$ $(0\bar{h}l)$ $(0hl)$ $(\bar{h}0l)$ $(h0l)$ $(0hl)$ $(0hl)$
4	<i>c</i>	$\dots m$	Tetragonal disphenoid or tetragonal tetrahedron <i>Tetragonal tetrahedron (n)</i> Tetragonal prism <i>Square through origin</i>	(hhl) $(\bar{h}\bar{h}l)$ $(h\bar{h}l)$ $(\bar{h}hl)$ (110) $(\bar{1}\bar{1}0)$ $(1\bar{1}0)$ $(\bar{1}10)$
4	<i>b</i>	$\dots 2$	Tetragonal prism <i>Square through origin (i)</i>	(100) $(\bar{1}00)$ $(0\bar{1}0)$ (010)
2	<i>a</i>	$2mm$	Pinacoid or parallelohedron <i>Line segment through origin (g)</i> Symmetry of special projections Along $[001]$ Along $[100]$ Along $[110]$ $4mm$ $2mm$ m	(001) $(00\bar{1})$
$\bar{4}m2$	D_{2d}			
8	<i>d</i>	1	Tetragonal scalenohedron <i>Tetragonal tetrahedron cut off by pinacoid (l)</i> Ditetragonal prism <i>Truncated square through origin</i> Tetragonal dipyramid <i>Tetragonal prism</i>	(hkl) $(\bar{h}\bar{k}l)$ $(k\bar{h}l)$ $(\bar{k}hl)$ $(h\bar{k}l)$ (hkl) $(\bar{k}hl)$ (khl) $(hk0)$ $(\bar{h}\bar{k}0)$ $(k\bar{h}0)$ $(\bar{k}h0)$ $(h\bar{k}0)$ $(hk0)$ $(\bar{k}h0)$ $(kh0)$ (hhl) $(\bar{h}\bar{h}l)$ $(h\bar{h}l)$ $(\bar{h}hl)$ $(\bar{h}hl)$ (hhl) (hhl) $(\bar{h}hl)$
4	<i>c</i>	$\dots m$	Tetragonal disphenoid or tetragonal tetrahedron <i>Tetragonal tetrahedron (j)</i> Tetragonal prism <i>Square through origin</i>	$(h0l)$ $(\bar{h}0l)$ $(0\bar{h}l)$ $(0hl)$ (100) $(\bar{1}00)$ $(0\bar{1}0)$ (010)
4	<i>b</i>	$\dots 2$	Tetragonal prism <i>Square through origin (h)</i>	(110) $(\bar{1}\bar{1}0)$ $(1\bar{1}0)$ $(\bar{1}10)$
2	<i>a</i>	$2mm$	Pinacoid or parallelohedron <i>Line segment through origin (e)</i> Symmetry of special projections Along $[001]$ Along $[100]$ Along $[110]$ $4mm$ m $2mm$	(001) $(00\bar{1})$

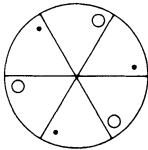
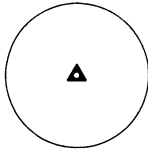
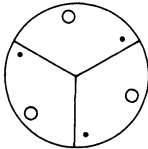
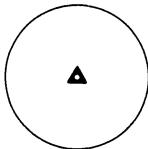
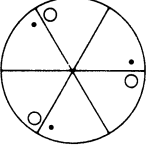
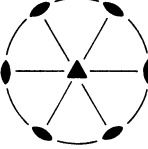
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. The 32 three-dimensional crystallographic point groups (cont.)

TETRAGONAL SYSTEM (cont.)					
$4/mmm$	D_{4h}				
$4 \ 2 \ 2$					
$m \ m \ m$					
16	g	1	Ditetragonal dipyramid <i>Edge-truncated tetragonal prism (u)</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{k}hl)$ $(k\bar{h}l)$ $(hk\bar{l})$ $(h\bar{k}\bar{l})$ (khl) $(\bar{k}\bar{h}\bar{l})$ $(\bar{h}k\bar{l})$ $(h\bar{k}l)$ $(k\bar{h}l)$ $(\bar{k}hl)$ $(\bar{h}kl)$ $(\bar{h}kl)$ $(\bar{k}hl)$ (khl)	
8	f	$.m.$	Tetragonal dipyramid <i>Tetragonal prism (s)</i>	$(h0l)$ $(\bar{h}0l)$ $(0hl)$ $(0\bar{h}l)$ $(h0\bar{l})$ $(h0l)$ $(0h\bar{l})$ $(0hl)$	
8	e	$..m$	Tetragonal dipyramid <i>Tetragonal prism (r)</i>	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}hl)$ $(h\bar{h}l)$ $(hh\bar{l})$ $(hh\bar{l})$ $(hh\bar{l})$ $(hh\bar{l})$	
8	d	$m..$	Ditetragonal prism <i>Truncated square through origin (p)</i>	$(hk0)$ $(\bar{h}\bar{k}0)$ $(\bar{k}h0)$ $(k\bar{h}0)$ $(\bar{h}k0)$ $(hk0)$ $(kh0)$ $(\bar{k}h0)$	
4	c	$m2m.$	Tetragonal prism <i>Square through origin (l)</i>	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$	
4	b	$m.m2$	Tetragonal prism <i>Square through origin (j)</i>	(110) $(\bar{1}\bar{1}0)$ $(\bar{1}10)$ $(1\bar{1}0)$	
2	a	$4mm$	Pinacoid or parallelohedron <i>Line segment through origin (g)</i>	(001) $(00\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[110]$		
			$4mm$ $2mm$ $2mm$		
TRIGONAL SYSTEM					
3	C_3				
HEXAGONAL AXES					
3	b	1	Trigonal pyramid <i>Trigon (d)</i>	(hki) (ihk) (kih)	
			Trigonal prism <i>Trigon through origin</i>	$(hki0)$ $(ihk0)$ $(kih0)$	
1	a	3..	Pedion or monohedron <i>Single point (a)</i>	(0001) or $(000\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[210]$		
			3 1 1		
3	C_3				
RHOMBOHEDRAL AXES					
3	b	1	Trigonal pyramid <i>Trigon (b)</i>	(hkl) (lkh) (klh)	
			Trigonal prism <i>Trigon through origin</i>	$(hk(\bar{h}+\bar{k}))$ $((\bar{h}+\bar{k})hk)$ $(k(\bar{h}+\bar{k})h)$	
1	a	3.	Pedion or monohedron <i>Single point (a)</i>	(111) or $(\bar{1}\bar{1}\bar{1})$	
Symmetry of special projections					
			Along $[111]$ Along $[1\bar{1}0]$ Along $[2\bar{1}\bar{1}]$		
			3 1 1		

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TRIGONAL SYSTEM (<i>cont.</i>)					
$\bar{3}$	C_{3i}				
HEXAGONAL AXES					
6	<i>b</i>	1	Rhombohedron <i>Trigonal antiprism (g)</i>	(hkl) (ihk) (kih) $(\bar{h}\bar{k}\bar{l})$ $(\bar{i}\bar{h}\bar{k})$ $(\bar{k}\bar{i}\bar{h})$	
			Hexagonal prism <i>Hexagon through origin</i>	$(hki0)$ $(ihk0)$ $(kih0)$ $(\bar{h}\bar{k}\bar{i}0)$ $(\bar{i}\bar{h}\bar{k}0)$ $(\bar{k}\bar{i}\bar{h}0)$	
2	<i>a</i>	3..	Pinacoid or parallelohedron <i>Line segment through origin (c)</i>	(0001) $(000\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[210]$		
			6 2 2		
RHOMBOHEDRAL AXES					
$\bar{3}$	C_{3i}				
6	<i>b</i>	1	Rhombohedron <i>Trigonal antiprism (f)</i>	(hkl) (lkh) (klh) $(\bar{h}\bar{k}\bar{l})$ $(\bar{l}\bar{h}\bar{k})$ $(\bar{k}\bar{l}\bar{h})$	
			Hexagonal prism <i>Hexagon through origin</i>	$(hk(\bar{h}+\bar{k}))$ $((\bar{h}+\bar{k})hk)$ $(k(\bar{h}+\bar{k})h)$ $(\bar{h}\bar{k}(h+k))$ $((h+k)\bar{h}\bar{k})$ $(\bar{k}(h+k)\bar{h})$	
2	<i>a</i>	3.	Pinacoid or parallelohedron <i>Line segment through origin (c)</i>	(111) $(\bar{1}\bar{1}\bar{1})$	
Symmetry of special projections					
			Along $[111]$ Along $[1\bar{1}0]$ Along $[2\bar{1}\bar{1}]$		
			6 2 2		
HEXAGONAL SYSTEM					
321	D_3				
6	<i>c</i>	1	Trigonal trapezohedron <i>Twisted trigonal antiprism (g)</i>	(hkl) (ihk) (kih) $(khi\bar{l})$ $(hik\bar{l})$ $(ikh\bar{l})$	
			Ditrigonal prism <i>Truncated trigon through origin</i>	$(hki0)$ $(ihk0)$ $(kih0)$ $(khi0)$ $(hik0)$ $(ikh0)$	
			Trigonal dipyramid <i>Trigonal prism</i>	$(hh\bar{2}hl)$ $(\bar{2}h\bar{h}hl)$ $(h\bar{2}hhl)$ $(hh2h\bar{l})$ $(h2h\bar{h}\bar{l})$ $(2h\bar{h}\bar{h}\bar{l})$	
			Rhombohedron <i>Trigonal antiprism</i>	$(h0hl)$ $(\bar{h}h0l)$ $(0\bar{h}hl)$ $(0h\bar{h}l)$ $(h\bar{h}0\bar{l})$ $(\bar{h}0h\bar{l})$	
			Hexagonal prism <i>Hexagon through origin</i>	$(10\bar{1}0)$ $(\bar{1}100)$ $(0\bar{1}10)$ $(01\bar{1}0)$ $(1\bar{1}00)$ $(\bar{1}010)$	
3	<i>b</i>	.2.	Trigonal prism <i>Trigon through origin (e)</i>	(1120) $(\bar{2}110)$ $(1\bar{2}10)$ or $(\bar{1}\bar{1}20)$ $(2\bar{1}\bar{1}0)$ $(\bar{1}2\bar{1}0)$	
2	<i>a</i>	3..	Pinacoid or parallelohedron <i>Line segment through origin (c)</i>	(0001) $(000\bar{1})$	
Symmetry of special projections					
			Along $[001]$ Along $[100]$ Along $[210]$		
			3m 2 1		

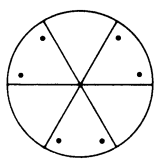
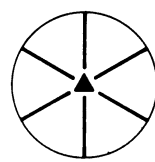
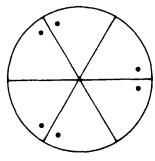
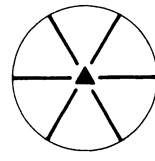
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. The 32 three-dimensional crystallographic point groups (cont.)

TRIGONAL SYSTEM (cont.)						
312 D_3						
HEXAGONAL AXES						
6	c	1	Trigonal trapezohedron <i>Twisted trigonal antiprism (l)</i>	(hkl) $(\bar{k}h\bar{l})$	(ihk) $(\bar{h}i\bar{k})$	$(kihl)$ $(\bar{i}k\bar{h}\bar{l})$
			Ditrigonal prism <i>Truncated trigon through origin</i>	$(hki0)$ $(\bar{k}h\bar{i}0)$	$(ihk0)$ $(\bar{h}i\bar{k}0)$	$(kih0)$ $(\bar{i}k\bar{h}0)$
			Trigonal dipyramid <i>Trigonal prism</i>	$(h0\bar{h}l)$ $(0h\bar{h}l)$	$(\bar{h}h0l)$ $(\bar{h}h0\bar{l})$	$(0\bar{h}hl)$ $(h0hl)$
			Rhombohedron <i>Trigonal antiprism</i>	$(hh2\bar{h}l)$ $(\bar{h}h2h\bar{l})$	$(\bar{2}hhhl)$ $(\bar{h}2h\bar{h}\bar{l})$	$(h2\bar{h}hl)$ $(2h\bar{h}\bar{h}\bar{l})$
			Hexagonal prism <i>Hexagon through origin</i>	$(11\bar{2}0)$ $(\bar{1}\bar{1}20)$	$(\bar{2}110)$ $(\bar{1}2\bar{1}0)$	$(1\bar{2}10)$ $(2\bar{1}\bar{1}0)$
3	b	..2	Trigonal prism <i>Trigon through origin (j)</i>	$(10\bar{1}0)$ or (1010)	$(\bar{1}100)$ $(\bar{1}\bar{1}00)$	$(0\bar{1}10)$ $(01\bar{1}0)$
2	a	3..	Pinacoid or parallelohedron <i>Line segment through origin (g)</i>	(0001)	$(000\bar{1})$	
			Symmetry of special projections			
			Along $[001]$	Along $[100]$	Along $[210]$	
			3m	1	2	
32 D_3						
RHOMBOHEDRAL AXES						
6	c	1	Trigonal trapezohedron <i>Twisted trigonal antiprism (f)</i>	(hkl) $(\bar{k}h\bar{l})$	(lkh) $(\bar{h}l\bar{k})$	(klh) $(\bar{l}k\bar{h})$
			Ditrigonal prism <i>Truncated trigon through origin</i>	$(hk(\bar{h}+\bar{k}))$ $(\bar{k}h(h+k))$	$((\bar{h}+\bar{k})hk)$ $(\bar{h}(h+k)\bar{k})$	$(k(\bar{h}+\bar{k})h)$ $((h+k)\bar{k}h)$
			Trigonal dipyramid <i>Trigonal prism</i>	$(hk(2k-h))$ $(\bar{k}h(h-2k))$	$((2k-h)hk)$ $(\bar{h}(h-2k)\bar{k})$	$(k(2k-h)h)$ $((h-2k)\bar{k}h)$
			Rhombohedron <i>Trigonal antiprism</i>	(hhl) $(\bar{h}h\bar{l})$	(lhh) $(\bar{h}l\bar{h})$	(hlh) $(\bar{l}h\bar{h})$
			Hexagonal prism <i>Hexagon through origin</i>	$(11\bar{2})$ $(\bar{1}\bar{1}2)$	$(\bar{2}11)$ $(\bar{1}2\bar{1})$	$(1\bar{2}1)$ $(2\bar{1}\bar{1})$
3	b	.2	Trigonal prism <i>Trigon through origin (d)</i>	$(01\bar{1})$ or $(0\bar{1}1)$	$(\bar{1}01)$ $(10\bar{1})$	$(1\bar{1}0)$ $(\bar{1}10)$
2	a	3.	Pinacoid or parallelohedron <i>Line segment through origin (c)</i>	(111)	$(\bar{1}\bar{1}\bar{1})$	
			Symmetry of special projections			
			Along $[111]$	Along $[1\bar{1}0]$	Along $[2\bar{1}\bar{1}]$	
			3m	2	1	

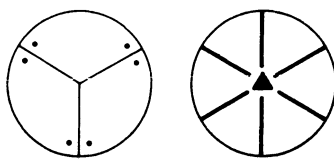
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

 Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TRIGONAL SYSTEM (<i>cont.</i>)								
3m1 C_{3v}								
HEXAGONAL AXES								
								
6	<i>c</i>	1	Ditrigonal pyramid <i>Truncated trigon (e)</i>			(hkl)	$(ihkl)$	$(kihl)$
						$(\bar{k}h\bar{l})$	$(\bar{h}i\bar{k}l)$	$(i\bar{k}h\bar{l})$
			Ditrigonal prism <i>Truncated trigon through origin</i>			$(hki0)$	$(ihk0)$	$(kih0)$
						$(\bar{k}h\bar{i}0)$	$(\bar{h}i\bar{k}0)$	$(i\bar{k}h0)$
			Hexagonal pyramid <i>Hexagon</i>			$(hh2hl)$	$(\bar{2}hhhl)$	$(h2\bar{h}hl)$
						$(\bar{h}h2hl)$	$(h2\bar{h}hl)$	$(2h\bar{h}hl)$
			Hexagonal prism <i>Hexagon through origin</i>			$(11\bar{2}0)$	$(\bar{2}110)$	$(1\bar{2}10)$
						$(\bar{1}120)$	$(1\bar{2}10)$	$(2\bar{1}10)$
3	<i>b</i>	<i>.m</i>	Trigonal pyramid <i>Trigon (d)</i>			$(h0\bar{h}l)$	$(\bar{h}h0l)$	$(0\bar{h}hl)$
			Trigonal prism <i>Trigon through origin</i>			$(10\bar{1}0)$	$(\bar{1}100)$	$(0\bar{1}10)$
						or $(\bar{1}010)$	$(1\bar{1}00)$	$(01\bar{1}0)$
1	<i>a</i>	<i>3m</i>	Pedion or monohedron <i>Single point (a)</i>			(0001) or $(000\bar{1})$		
			Symmetry of special projections					
			Along $[001]$	Along $[100]$	Along $[210]$			
			$3m$	1	m			
31m C_{3v}								
HEXAGONAL AXES								
								
6	<i>c</i>	1	Ditrigonal pyramid <i>Truncated trigon (d)</i>			(hkl)	$(ihkl)$	$(kihl)$
						(khl)	$(hikl)$	$(ikh\bar{l})$
			Ditrigonal prism <i>Truncated trigon through origin</i>			$(hki0)$	$(ihk0)$	$(kih0)$
						$(khi0)$	$(hik0)$	$(ikh0)$
			Hexagonal pyramid <i>Hexagon</i>			$(h0\bar{h}l)$	$(\bar{h}h0l)$	$(0\bar{h}hl)$
						$(0h\bar{h}l)$	$(h\bar{h}0l)$	$(h0hl)$
			Hexagonal prism <i>Hexagon through origin</i>			$(10\bar{1}0)$	$(\bar{1}100)$	$(0\bar{1}10)$
						$(01\bar{1}0)$	$(1\bar{1}00)$	$(\bar{1}010)$
3	<i>b</i>	<i>.m</i>	Trigonal pyramid <i>Trigon (c)</i>			$(hh2\bar{h}l)$	$(\bar{2}hhhl)$	$(h2\bar{h}hl)$
			Trigonal prism <i>Trigon through origin</i>			$(11\bar{2}0)$	$(\bar{2}110)$	$(1\bar{2}10)$
						or $(\bar{1}\bar{1}20)$	$(2\bar{1}\bar{1}0)$	$(\bar{1}2\bar{1}0)$
1	<i>a</i>	<i>3m</i>	Pedion or monohedron <i>Single point (a)</i>			(0001) or $(000\bar{1})$		
			Symmetry of special projections					
			Along $[001]$	Along $[100]$	Along $[210]$			
			$3m$	m	1			

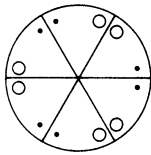
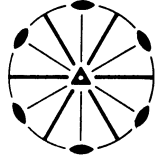
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. The 32 three-dimensional crystallographic point groups (cont.)

TRIGONAL SYSTEM (cont.)				
$3m$	C_{3v}			
			RHOMBOHEDRAL AXES	
6	c	1	Ditrigonal pyramid <i>Truncated trigon (c)</i>	$(hkl) (lkh) (klh)$ $(khl) (hlk) (lkh)$
			Ditrigonal prism <i>Truncated trigon through origin</i>	$(hk(\bar{h}+\bar{k})) ((\bar{h}+\bar{k})hk) (k(\bar{h}+\bar{k})h)$ $(kh(\bar{h}+\bar{k})) (h(\bar{h}+\bar{k})k) ((\bar{h}+\bar{k})kh)$
			Hexagonal pyramid <i>Hexagon</i>	$(hk(2k-h)) ((2k-h)hk) (k(2k-h)h)$ $(kh(2k-h)) (h(2k-h)k) ((2k-h)kh)$
			Hexagonal prism <i>Hexagon through origin</i>	$(0\bar{1}\bar{1}) (\bar{1}0\bar{1}) (\bar{1}\bar{1}0)$ $(10\bar{1}) (0\bar{1}1) (\bar{1}10)$
3	b	$.m$	Trigonal pyramid <i>Trigon (b)</i>	$(hhl) (lhh) (hlh)$
			Trigonal prism <i>Trigon through origin</i>	$(11\bar{2}) (\bar{2}11) (\bar{1}\bar{2}1)$ or $(\bar{1}\bar{1}2) (2\bar{1}\bar{1}) (\bar{1}2\bar{1})$
1	a	$3m$	Pedion or monohedron <i>Single point (a)</i>	(111) or $(\bar{1}\bar{1}\bar{1})$
			Symmetry of special projections	
			Along $[111]$ Along $[1\bar{1}0]$ Along $[2\bar{1}\bar{1}]$	
			$3m$ 1 m	
$\bar{3}m1$	D_{3d}			
$\bar{3}21$			HEXAGONAL AXES	
			m	
12	d	1	Ditrigonal scalenohedron or hexagonal scalenohedron <i>Trigonal antiprism sliced off by pinacoid (j)</i>	$(hki) (ihk) (kih)$ $(khi) (hik) (ikh)$ $(\bar{h}\bar{k}\bar{i}) (\bar{i}\bar{h}\bar{k}) (\bar{k}\bar{i}\bar{h})$ $(\bar{k}\bar{h}\bar{i}) (\bar{h}\bar{i}\bar{k}) (\bar{i}\bar{k}\bar{h})$
			Dihexagonal prism <i>Truncated hexagon through origin</i>	$(hki0) (ihk0) (kih0)$ $(khi0) (hik0) (ikh0)$ $(\bar{h}\bar{k}\bar{i}0) (\bar{i}\bar{h}\bar{k}0) (\bar{k}\bar{i}\bar{h}0)$ $(\bar{k}\bar{h}\bar{i}0) (\bar{h}\bar{i}\bar{k}0) (\bar{i}\bar{k}\bar{h}0)$
			Hexagonal dipyrmaid <i>Hexagonal prism</i>	$(h\bar{h}2hl) (\bar{2}h\bar{h}hl) (h\bar{2}hhl)$ $(hh\bar{2}hl) (\bar{h}2\bar{h}hl) (\bar{2}hhhl)$ $(\bar{h}\bar{h}2hl) (2h\bar{h}hl) (\bar{h}2hhl)$ $(\bar{h}\bar{h}2hl) (\bar{h}2hhl) (2h\bar{h}hl)$
6	c	$.m$	Rhombohedron <i>Trigonal antiprism (i)</i>	$(h0hl) (\bar{h}h0l) (0\bar{h}hl)$ $(0hh\bar{l}) (\bar{h}h0\bar{l}) (\bar{h}0h\bar{l})$
			Hexagonal prism <i>Hexagon through origin</i>	$(10\bar{1}0) (\bar{1}100) (0\bar{1}10)$ $(01\bar{1}0) (\bar{1}\bar{1}00) (\bar{1}010)$
6	b	$.2$	Hexagonal prism <i>Hexagon through origin (g)</i>	$(11\bar{2}0) (\bar{2}110) (1\bar{2}10)$ $(\bar{1}\bar{1}20) (\bar{1}\bar{2}10) (2\bar{1}\bar{1}0)$
2	a	$3m$	Pinacoid or parallelohedron <i>Line segment through origin (c)</i>	$(0001) (000\bar{1})$
			Symmetry of special projections	
			Along $[001]$ Along $[100]$ Along $[210]$	
			$6mm$ 2 $2mm$	

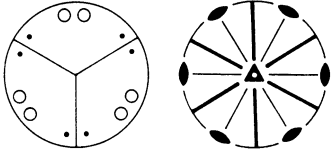
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TRIGONAL SYSTEM (<i>cont.</i>)						
$\bar{3}1m$	D_{3d}					
$\bar{3}1\frac{2}{m}$						
HEXAGONAL AXES						
12	<i>d</i>	1	Ditrigonal scalenohedron or hexagonal scalenohedron <i>Trigonal antiprism sliced off by pinacoid (l)</i>	(hkl) $(\bar{k}h\bar{l})$ $(\bar{h}k\bar{l})$ (khl)	$(ihkl)$ $(\bar{h}ikl)$ $(\bar{i}hkl)$ $(hikl)$	$(kihl)$ $(\bar{i}khl)$ $(\bar{k}ihl)$ $(ikhil)$
			Dihexagonal prism <i>Truncated hexagon through origin</i>	$(hki0)$ $(\bar{k}h\bar{i}0)$ $(\bar{h}k\bar{i}0)$ $(khi0)$	$(ihk0)$ $(\bar{h}ik0)$ $(\bar{i}hk0)$ $(hik0)$	$(kih0)$ $(\bar{i}k\bar{h}0)$ $(\bar{k}ih0)$ $(ikh0)$
			Hexagonal dipyramid <i>Hexagonal prism</i>	$(h0\bar{h}l)$ $(0\bar{h}hl)$ $(h0hl)$ $(0hhl)$	$(\bar{h}h0l)$ $(\bar{h}h0\bar{l})$ $(h\bar{h}0l)$ $(h\bar{h}0\bar{l})$	$(0\bar{h}hl)$ $(h0\bar{h}l)$ $(0hhl)$ $(h0hl)$
6	<i>c</i>	$\dots m$	Rhombohedron <i>Trigonal antiprism (k)</i>	$(hh2\bar{h}l)$ $(\bar{h}\bar{h}2hl)$	$(\bar{2}h\bar{h}hl)$ $(\bar{h}2hhl)$	$(h2\bar{h}hl)$ $(2h\bar{h}hl)$
			Hexagonal prism <i>Hexagon through origin</i>	$(11\bar{2}0)$ $(\bar{1}\bar{1}20)$	$(\bar{2}110)$ $(\bar{1}2\bar{1}0)$	$(1\bar{2}10)$ $(2\bar{1}\bar{1}0)$
6	<i>b</i>	$\dots 2$	Hexagonal prism <i>Hexagon through origin (i)</i>	$(10\bar{1}0)$ $(\bar{1}010)$	$(\bar{1}100)$ $(1\bar{1}00)$	$(0\bar{1}10)$ $(01\bar{1}0)$
2	<i>a</i>	$3.m$	Pinacoid or parallelohedron <i>Line segment through origin (e)</i>	(0001)	$(000\bar{1})$	
			Symmetry of special projections			
			Along [001]	Along [100]	Along [210]	
			$6mm$	$2mm$	2	

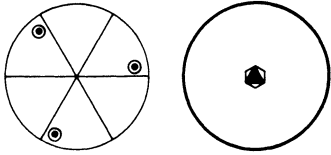
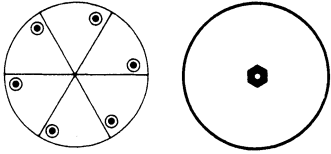
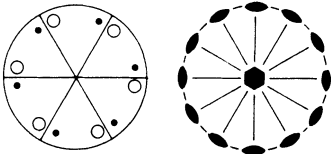
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

TRIGONAL SYSTEM (cont.)									
$\bar{3}m$	D_{3d}								
$\bar{3} \frac{2}{m}$									
RHOMBOHEDRAL AXES									
12	d	1	Ditrigonal scalenohedron or hexagonal scalenohedron <i>Trigonal antiprism sliced off by pinacoid (i)</i>	(hkl) $(\bar{k}h\bar{l})$ $(h\bar{k}l)$ (khl)	(lhk) $(\bar{h}l\bar{k})$ $(l\bar{h}k)$ (hlk)	(klh) $(\bar{l}k\bar{h})$ $(k\bar{l}h)$ (lkh)			
			Dihexagonal prism <i>Truncated hexagon through origin</i>	$(hk(\bar{h}+\bar{k}))$ $(\bar{k}h(h+k))$ $(\bar{h}\bar{k}(h+k))$ $(kh(\bar{h}+\bar{k}))$	$((\bar{h}+\bar{k})hk)$ $(\bar{h}(h+k)\bar{k})$ $((h+k)\bar{h}\bar{k})$ $(h(\bar{h}+\bar{k})k)$	$(k(\bar{h}+\bar{k})h)$ $((h+k)\bar{k}\bar{h})$ $(\bar{k}(h+k)\bar{h})$ $((\bar{h}+\bar{k})kh)$			
			Hexagonal dipyramid <i>Hexagonal prism</i>	$(hk(2k-h))$ $(\bar{k}h(h-2k))$ $(\bar{h}\bar{k}(h-2k))$ $(kh(2k-h))$	$((2k-h)hk)$ $(\bar{h}(h-2k)\bar{k})$ $((h-2k)\bar{h}\bar{k})$ $(h(2k-h)k)$	$(k(2k-h)h)$ $((h-2k)\bar{k}\bar{h})$ $(\bar{k}(h-2k)\bar{h})$ $((2k-h)kh)$			
6	c	$.m$	Rhombohedron <i>Trigonal antiprism (h)</i> Hexagonal prism <i>Hexagon through origin</i>	(hhl) $(\bar{h}\bar{h}l)$	(lhh) $(\bar{l}\bar{h}\bar{h})$	(hlh) $(\bar{l}\bar{h}\bar{h})$			
6	b	$.2$	Hexagonal prism <i>Hexagon through origin (f)</i>	$(01\bar{1})$ $(0\bar{1}1)$	$(\bar{1}01)$ $(10\bar{1})$	$(1\bar{1}0)$ $(\bar{1}10)$			
2	a	$3m$	Pinacoid or parallelohedron <i>Line segment through origin (c)</i> Symmetry of special projections Along $[111]$ Along $[\bar{1}\bar{1}0]$ Along $[2\bar{1}\bar{1}]$ $6mm$ 2 $2mm$	(111)	$(\bar{1}\bar{1}\bar{1})$				
HEXAGONAL SYSTEM									
6	C_6								
6	b	1	Hexagonal pyramid <i>Hexagon (d)</i> Hexagonal prism <i>Hexagon through origin</i>	(hki) $(hki0)$	(ihk) $(ihk0)$	(kih) $(kih0)$	$(\bar{h}\bar{k}\bar{i})$ $(\bar{h}\bar{k}\bar{i}0)$	$(\bar{i}\bar{h}\bar{k})$ $(\bar{i}\bar{h}\bar{k}0)$	$(\bar{k}\bar{i}\bar{h})$ $(\bar{k}\bar{i}\bar{h}0)$
1	a	6..	Pedion or monohedron <i>Single point (a)</i> Symmetry of special projections Along $[001]$ Along $[100]$ Along $[210]$ 6 m m	(0001)	or $(000\bar{1})$				

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. The 32 three-dimensional crystallographic point groups (cont.)

HEXAGONAL SYSTEM (cont.)						
$\bar{6}$	C_{3h}					
6	<i>c</i>	1	Trigonal dipyramid Trigonal prism (<i>l</i>)		(hkl) $(ihkl)$ $(kihl)$ $(h\bar{k}l)$ $(ih\bar{k}l)$ $(k\bar{i}hl)$	
3	<i>b</i>	<i>m.</i>	Trigonal prism Trigonal through origin (<i>j</i>)		$(hki0)$ $(ihk0)$ $(kih0)$	
2	<i>a</i>	3..	Pinacoid or parallelohedron Line segment through origin (<i>g</i>)		(0001) $(000\bar{1})$	
			Symmetry of special projections			
			Along $[001]$	Along $[100]$	Along $[210]$	
			3	<i>m</i>	<i>m</i>	
$6/m$	C_{6h}					
12	<i>c</i>	1	Hexagonal dipyramid Hexagonal prism (<i>l</i>)		(hkl) $(ihkl)$ $(kihl)$ $(h\bar{k}l)$ $(ih\bar{k}l)$ $(k\bar{i}hl)$ $(h\bar{k}\bar{l})$ $(ih\bar{k}\bar{l})$ $(k\bar{i}\bar{h}l)$	
6	<i>b</i>	<i>m.</i>	Hexagonal prism Hexagonal through origin (<i>j</i>)		$(hki0)$ $(ihk0)$ $(kih0)$ $(h\bar{k}i0)$ $(ih\bar{k}0)$ $(k\bar{i}h0)$	
2	<i>a</i>	6..	Pinacoid or parallelohedron Line segment through origin (<i>e</i>)		(0001) $(000\bar{1})$	
			Symmetry of special projections			
			Along $[001]$	Along $[100]$	Along $[210]$	
			6	$2mm$	$2mm$	
622	D_6					
12	<i>d</i>	1	Hexagonal trapezohedron Twisted hexagonal antiprism (<i>n</i>) Dihexagonal prism Truncated hexagon through origin		(hkl) $(ihkl)$ $(kihl)$ $(h\bar{k}l)$ $(ih\bar{k}l)$ $(k\bar{i}hl)$ $(kh\bar{l})$ $(hik\bar{l})$ $(k\bar{i}h\bar{l})$ $(h\bar{k}\bar{l})$ $(ih\bar{k}\bar{l})$ $(k\bar{i}\bar{h}\bar{l})$ $(hki0)$ $(ihk0)$ $(kih0)$ $(h\bar{k}i0)$ $(ih\bar{k}0)$ $(k\bar{i}h0)$ $(khi0)$ $(hik0)$ $(ikh0)$ $(k\bar{h}i0)$ $(h\bar{i}k0)$ $(i\bar{k}h0)$	
			Hexagonal dipyramid Hexagonal prism		$(h0hl)$ $(h0hl)$ $(0\bar{h}hl)$ $(\bar{h}0hl)$ $(h\bar{h}0l)$ $(0hhl)$ $(0hhl)$ $(h\bar{h}0l)$ $(\bar{h}0hl)$ $(0\bar{h}hl)$ $(h\bar{h}0l)$ $(0hhl)$	
			Hexagonal dipyramid Hexagonal prism		$(hh2hl)$ $(2h\bar{h}hl)$ $(h\bar{h}2hl)$ $(\bar{h}h2hl)$ $(2hh\bar{h}l)$ $(\bar{h}2h\bar{h}l)$ $(hh2hl)$ $(h2h\bar{h}l)$ $(2h\bar{h}hl)$ $(\bar{h}h2hl)$ $(\bar{h}2h\bar{h}l)$ $(2hh\bar{h}l)$	
6	<i>c</i>	..2	Hexagonal prism Hexagonal through origin (<i>l</i>)		$(10\bar{1}0)$ $(\bar{1}100)$ $(0\bar{1}10)$ $(\bar{1}010)$ $(1\bar{1}00)$ $(01\bar{1}0)$	
6	<i>b</i>	.2.	Hexagonal prism Hexagonal through origin (<i>j</i>)		$(11\bar{2}0)$ $(\bar{2}110)$ $(1\bar{2}10)$ $(\bar{1}\bar{1}20)$ $(2\bar{1}\bar{1}0)$ $(\bar{1}2\bar{1}0)$	
2	<i>a</i>	6..	Pinacoid or parallelohedron Line segment through origin (<i>e</i>)		(0001) $(000\bar{1})$	
			Symmetry of special projections			
			Along $[001]$	Along $[100]$	Along $[210]$	
			$6mm$	$2mm$	$2mm$	

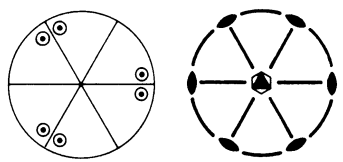
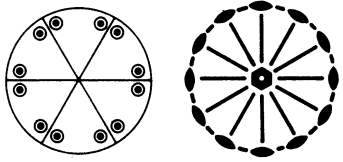
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

HEXAGONAL SYSTEM (cont.)									
$6mm$	C_{6v}								
12	d	1	Dihexagonal pyramid Truncated hexagon (f)	(hki) (khi)	$(ihkl)$ $(hikl)$	$(kihl)$ (ikh)	$(\bar{h}\bar{k}\bar{l})$ $(\bar{k}\bar{h}\bar{l})$	$(\bar{i}\bar{h}\bar{k}\bar{l})$ $(\bar{h}\bar{i}\bar{k}\bar{l})$	$(\bar{k}\bar{i}\bar{h}\bar{l})$ $(\bar{i}\bar{k}\bar{h}\bar{l})$
			Dihexagonal prism Truncated hexagon through origin	$(hki0)$ $(khi0)$	$(ihk0)$ $(hik0)$	$(kih0)$ $(ikh0)$	$(\bar{h}\bar{k}\bar{i}0)$ $(\bar{k}\bar{h}\bar{i}0)$	$(\bar{i}\bar{h}\bar{k}0)$ $(\bar{h}\bar{i}\bar{k}0)$	$(\bar{k}\bar{i}\bar{h}0)$ $(\bar{i}\bar{k}\bar{h}0)$
6	c	$.m$	Hexagonal pyramid Hexagon (e)	$(h0\bar{h}l)$	$(\bar{h}h0l)$	$(0\bar{h}hl)$	$(\bar{h}0hl)$	$(h\bar{h}0l)$	$(0h\bar{h}l)$
			Hexagonal prism Hexagon through origin	$(10\bar{1}0)$	$(\bar{1}100)$	$(0\bar{1}10)$	$(\bar{1}010)$	$(1\bar{1}00)$	$(01\bar{1}0)$
6	b	$.m$	Hexagonal pyramid Hexagon (d)	$(hh2\bar{h}l)$	$(\bar{2}hhhl)$	$(h2\bar{h}hl)$	$(\bar{h}\bar{h}2hl)$	$(2h\bar{h}hl)$	$(\bar{h}2hhl)$
			Hexagonal prism Hexagon through origin	$(11\bar{2}0)$	$(\bar{2}110)$	$(1\bar{2}10)$	$(\bar{1}\bar{1}20)$	$(2\bar{1}\bar{1}0)$	$(\bar{1}2\bar{1}0)$
1	a	$6mm$	Pedion or monohedron Single point (a)	(0001) or $(000\bar{1})$					
			Symmetry of special projections						
			Along $[001]$	Along $[100]$	Along $[210]$				
			$6mm$	m	m				
$\bar{6}m2$	D_{3h}								
12	e	1	Ditrigonal dipyramid Edge-truncated trigonal prism (o)	(hki) (khi) $(\bar{k}\bar{h}\bar{l})$ $(\bar{k}\bar{h}\bar{l})$	$(ihkl)$ $(hikl)$ $(\bar{h}\bar{i}\bar{k}\bar{l})$ $(\bar{h}\bar{i}\bar{k}\bar{l})$	$(kihl)$ (ikh) $(\bar{i}\bar{k}\bar{h}\bar{l})$ $(\bar{i}\bar{k}\bar{h}\bar{l})$			
			Hexagonal dipyramid Hexagonal prism	$(hh2\bar{h}l)$ $(hh2hl)$ $(\bar{h}\bar{h}2hl)$ $(\bar{h}\bar{h}2hl)$	$(\bar{2}hhhl)$ $(2hhhl)$ $(\bar{h}2hhl)$ $(h2hhl)$	$(h2\bar{h}hl)$ $(h2hh\bar{l})$ $(2h\bar{h}hl)$ $(2h\bar{h}hl)$			
6	d	$m..$	Ditrigonal prism Truncated trigon through origin (l)	$(hki0)$ $(\bar{k}\bar{h}\bar{i}0)$	$(ihk0)$ $(\bar{h}\bar{i}\bar{k}0)$	$(kih0)$ $(\bar{i}\bar{k}\bar{h}0)$			
			Hexagonal prism Hexagon through origin	$(11\bar{2}0)$ $(\bar{1}\bar{1}20)$	$(\bar{2}110)$ $(1\bar{2}10)$	$(1\bar{2}10)$ $(2\bar{1}\bar{1}0)$			
6	c	$.m$	Trigonal dipyramid Trigonal prism (n)	$(h0\bar{h}l)$ $(h0\bar{h}\bar{l})$	$(\bar{h}h0l)$ $(\bar{h}h0\bar{l})$	$(0\bar{h}hl)$ $(0\bar{h}\bar{h}\bar{l})$			
3	b	$mm2$	Trigonal prism Trigon through origin (j)	$(10\bar{1}0)$ or $(\bar{1}010)$	$(\bar{1}100)$ $(1\bar{1}00)$	$(0\bar{1}10)$ $(01\bar{1}0)$			
2	a	$3m$	Pinacoid or parallelohedron Line segment through origin (g)	(0001) or $(000\bar{1})$					
			Symmetry of special projections						
			Along $[001]$	Along $[100]$	Along $[210]$				
			$3m$	m	$2mm$				

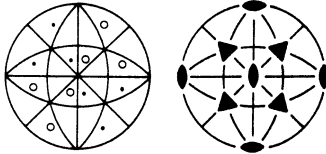
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

HEXAGONAL SYSTEM (cont.)					
$\bar{6}2m$	D_{3h}				
12	e	1	Ditrigonal dipyramid <i>Edge-truncated trigonal prism (l)</i> Hexagonal dipyramid <i>Hexagonal prism</i>	(hkl) $(ihkl)$ $(kihl)$ $(h\bar{k}l)$ $(i\bar{h}kl)$ $(k\bar{i}hl)$ $(kh\bar{l})$ $(hik\bar{l})$ $(ikh\bar{l})$ $(k\bar{h}\bar{l})$ $(\bar{h}i\bar{k}\bar{l})$ $(\bar{i}k\bar{h}\bar{l})$	
6	d	$m..$	Ditrigonal prism <i>Truncated trigon through origin (j)</i> Hexagonal prism <i>Hexagon through origin</i>	$(hk\bar{i}0)$ $(ihk0)$ $(kih0)$ $(k\bar{h}i0)$ $(hik0)$ $(ikh0)$ $(10\bar{1}0)$ $(\bar{1}100)$ $(0\bar{1}10)$ $(01\bar{1}0)$ $(1\bar{1}00)$ (1010)	
6	c	$..m$	Trigonal dipyramid <i>Trigonal prism (i)</i>	$(hh\bar{2}hl)$ $(\bar{2}hhhl)$ $(h\bar{2}hhl)$ $(hh2\bar{h}l)$ $(2hh\bar{h}l)$ $(h2\bar{h}hl)$	
3	b	$m2m$	Trigonal prism <i>Trigon through origin (f)</i>	$(11\bar{2}0)$ $(\bar{2}110)$ $(1\bar{2}10)$ or $(\bar{1}\bar{1}20)$ $(2\bar{1}\bar{1}0)$ $(\bar{1}2\bar{1}0)$	
2	a	$3.m$	Pinacoid or parallelohedron <i>Line segment through origin (e)</i>	(0001) $(000\bar{1})$	
			Symmetry of special projections		
			Along $[001]$	Along $[100]$	Along $[210]$
			$3m$	$2mm$	m
$6/mmm$	D_{6h}				
$\frac{6}{m} \frac{2}{m} \frac{2}{m}$					
24	g	1	Dihexagonal dipyramid <i>Edge-truncated hexagonal prism (r)</i>	(hkl) $(ihkl)$ $(kihl)$ $(\bar{h}\bar{k}\bar{l})$ $(\bar{i}\bar{h}\bar{k}\bar{l})$ $(\bar{k}\bar{i}\bar{h}\bar{l})$ $(kh\bar{l})$ $(hik\bar{l})$ $(ikh\bar{l})$ $(k\bar{h}\bar{i}\bar{l})$ $(\bar{h}\bar{i}\bar{k}\bar{l})$ $(\bar{i}\bar{k}\bar{h}\bar{l})$ $(\bar{h}\bar{k}\bar{l})$ $(\bar{i}\bar{h}\bar{k}\bar{l})$ $(\bar{k}\bar{i}\bar{h}\bar{l})$ $(h\bar{k}\bar{l})$ $(ih\bar{k}\bar{l})$ $(ki\bar{h}\bar{l})$ $(\bar{k}\bar{h}\bar{l})$ $(\bar{h}\bar{i}\bar{k}\bar{l})$ $(\bar{i}\bar{k}\bar{h}\bar{l})$ $(k\bar{h}l)$ $(hikl)$ $(ikh\bar{l})$	
12	f	$m..$	Dihexagonal prism <i>Truncated hexagon through origin (p)</i>	$(hk\bar{i}0)$ $(ihk0)$ $(kih0)$ $(\bar{h}\bar{k}\bar{i}0)$ $(\bar{i}\bar{h}\bar{k}0)$ $(\bar{k}\bar{i}\bar{h}0)$ $(k\bar{h}i0)$ $(hik0)$ $(ikh0)$ $(k\bar{h}\bar{i}0)$ $(\bar{h}\bar{i}\bar{k}0)$ $(\bar{i}\bar{k}\bar{h}0)$	
12	e	$.m.$	Hexagonal dipyramid <i>Hexagonal prism (o)</i>	$(h0\bar{h}l)$ $(\bar{h}h0l)$ $(0\bar{h}hl)$ $(\bar{h}0hl)$ $(h\bar{h}0l)$ $(0h\bar{h}l)$ $(0h\bar{h}l)$ $(h\bar{h}0\bar{l})$ $(\bar{h}0h\bar{l})$ $(0\bar{h}h\bar{l})$ $(h\bar{h}0\bar{l})$ $(h0\bar{h}\bar{l})$	
12	d	$..m$	Hexagonal dipyramid <i>Hexagonal prism (n)</i>	$(hh\bar{2}hl)$ $(\bar{2}hhhl)$ $(h\bar{2}hhl)$ $(\bar{h}\bar{2}hhl)$ $(2h\bar{h}hl)$ $(\bar{h}2hhl)$ $(hh2\bar{h}l)$ $(h\bar{2}hhl)$ $(\bar{2}hhhl)$ $(\bar{h}\bar{2}hhl)$ $(h2hhl)$ $(2h\bar{h}hl)$	
6	c	$mm2$	Hexagonal prism <i>Hexagon through origin (l)</i>	$(10\bar{1}0)$ $(\bar{1}100)$ $(0\bar{1}10)$ $(\bar{1}010)$ $(1\bar{1}00)$ $(01\bar{1}0)$	
6	b	$m2m$	Hexagonal prism <i>Hexagon through origin (j)</i>	$(11\bar{2}0)$ $(\bar{2}110)$ $(1\bar{2}10)$ $(\bar{1}\bar{1}20)$ $(2\bar{1}\bar{1}0)$ $(\bar{1}2\bar{1}0)$	
2	a	$6mm$	Pinacoid or parallelohedron <i>Line segment through origin (e)</i>	(0001) $(000\bar{1})$	
			Symmetry of special projections		
			Along $[001]$	Along $[100]$	Along $[210]$
			$6mm$	$2mm$	$2mm$

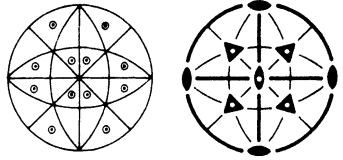
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

CUBIC SYSTEM						
23	<i>T</i>					
12	<i>c</i>	1	Pentagon-tritetrahedron or tetartoid or tetrahedral pentagon-dodecahedron <i>Snub tetrahedron</i> (= <i>pentagon-tritetrahedron</i> + <i>two tetrahedra</i>) (<i>j</i>)	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{h}k\bar{l})$ $(h\bar{k}\bar{l})$ (lhk) $(l\bar{h}\bar{k})$ $(l\bar{h}k)$ $(l\bar{h}\bar{k})$ (klh) $(\bar{k}\bar{l}h)$ $(k\bar{l}h)$ $(\bar{k}l\bar{h})$		
			$\left\{ \begin{array}{l} \text{Trigon-tritetrahedron} \\ \text{or tristetrahedron (for } h < l) \\ \text{Tetrahedron truncated by tetrahedron} \\ \text{(for } x < z) \end{array} \right\}$	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}h\bar{l})$ $(h\bar{h}\bar{l})$ (lhh) $(l\bar{h}\bar{h})$ $(l\bar{h}h)$ $(l\bar{h}\bar{h})$ (hll) $(\bar{h}l\bar{l})$ (hll) $(\bar{h}l\bar{l})$		
			$\left\{ \begin{array}{l} \text{Tetragon-tritetrahedron or deltohedron} \\ \text{or delto-dodecahedron (for } h > l) \\ \text{Cube \& two tetrahedra (for } x > z) \end{array} \right\}$			
			Pentagon-dodecahedron or dihexahedron or pyritohedron <i>Irregular icosahedron</i> (= <i>pentagon-dodecahedron</i> + <i>octahedron</i>)	$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$ $(l0k)$ $(l0\bar{k})$ $(l0k)$ $(l0\bar{k})$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(k\bar{l}0)$ $(\bar{k}l0)$		
			Rhomb-dodecahedron <i>Cuboctahedron</i>	(011) $(0\bar{1}1)$ $(01\bar{1})$ $(0\bar{1}\bar{1})$ (101) $(10\bar{1})$ (101) $(10\bar{1})$ (110) $(\bar{1}10)$ $(1\bar{1}0)$ $(\bar{1}\bar{1}0)$		
6	<i>b</i>	2..	Cube or hexahedron <i>Octahedron</i> (<i>f</i>)	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$ (001) $(00\bar{1})$		
4	<i>a</i>	.3.	Tetrahedron <i>Tetrahedron</i> (<i>e</i>)	(111) $(\bar{1}\bar{1}1)$ $(\bar{1}1\bar{1})$ $(1\bar{1}\bar{1})$ or $(\bar{1}\bar{1}\bar{1})$ (111) $(1\bar{1}\bar{1})$ $(\bar{1}\bar{1}1)$		
			Symmetry of special projections Along $[001]$ Along $[111]$ Along $[110]$ $2mm$ 3 m			

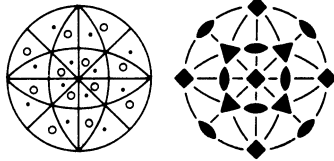
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

CUBIC SYSTEM (cont.)					
$m\bar{3}$	T_h				
$\frac{2}{m}\bar{3}$					
24	d	1	Didodecahedron or diploid or dyakisdodecahedron Cube & octahedron & pentagon-dodecahedron (l)	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{h}k\bar{l})$ $(h\bar{k}\bar{l})$ (lhk) $(l\bar{h}\bar{k})$ $(l\bar{h}k)$ $(lh\bar{k})$ (klh) $(\bar{k}\bar{l}h)$ $(\bar{k}lh)$ $(k\bar{l}\bar{h})$ $(\bar{h}\bar{k}l)$ $(h\bar{k}l)$ $(\bar{h}k\bar{l})$ $(h\bar{k}\bar{l})$ $(l\bar{h}k)$ $(lh\bar{k})$ (lhk) $(l\bar{h}\bar{k})$ $(\bar{k}lh)$ $(k\bar{l}h)$ $(\bar{k}lh)$ $(k\bar{l}\bar{h})$	
			{ Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron (for $ h < l $) Cube & octahedron & rhomb- dodecahedron (for $ x < z $) Trigon-trioctahedron or trisoctahedron (for $ h > l $) Cube truncated by octahedron (for $ x > z $)	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}h\bar{l})$ $(h\bar{h}\bar{l})$ (lhh) $(l\bar{h}\bar{h})$ $(l\bar{h}h)$ $(lh\bar{h})$ (hll) $(\bar{h}\bar{l}l)$ $(\bar{h}l\bar{l})$ $(hl\bar{l})$ $(\bar{h}\bar{h}l)$ $(h\bar{h}l)$ $(\bar{h}h\bar{l})$ $(h\bar{h}\bar{l})$ $(l\bar{h}h)$ $(lh\bar{h})$ (lhk) $(l\bar{h}\bar{k})$ $(\bar{h}l\bar{l})$ $(h\bar{l}l)$ $(\bar{h}l\bar{l})$ $(hl\bar{l})$	
12	c	$m..$	Pentagon-dodecahedron or dihexahedron or pyritohedron Irregular icosahedron (= pentagon-dodecahedron + octahedron) (j) Rhomb-dodecahedron Cuboctahedron	$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$ $(l0k)$ $(l0\bar{k})$ $(l0k)$ $(l0\bar{k})$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(k\bar{l}0)$ $(\bar{k}l0)$ (011) $(0\bar{1}\bar{1})$ $(01\bar{1})$ $(0\bar{1}1)$ (101) $(10\bar{1})$ (101) $(10\bar{1})$ (110) (110) (110) (110)	
8	b	.3.	Octahedron Cube (i)	(111) $(\bar{1}\bar{1}1)$ $(\bar{1}1\bar{1})$ $(1\bar{1}\bar{1})$ $(\bar{1}\bar{1}\bar{1})$ (111) (111) $(\bar{1}\bar{1}\bar{1})$	
6	a	$2mm..$	Cube or hexahedron Octahedron (e)	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$ (001) $(00\bar{1})$	
			Symmetry of special projections Along $[001]$ Along $[111]$ Along $[110]$ $2mm$ 6 $2mm$		

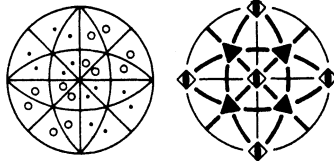
10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

CUBIC SYSTEM (<i>cont.</i>)					
432	<i>O</i>				
24	<i>d</i>	1	Pentagon-trioctahedron or gyroid or pentagon-icositetrahedron <i>Snub cube</i> (= <i>cube</i> + <i>octahedron</i> + <i>pentagon- trioctahedron</i>) (<i>k</i>)	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{h}k\bar{l})$ $(h\bar{k}\bar{l})$ $(kh\bar{l})$ $(\bar{k}\bar{h}l)$ $(k\bar{h}l)$ $(\bar{k}hl)$ (lhk) $(l\bar{h}\bar{k})$ $(l\bar{h}k)$ $(l\bar{h}\bar{k})$ (lkh) $(l\bar{k}\bar{h})$ $(lk\bar{h})$ $(l\bar{k}h)$ (klh) $(\bar{k}l\bar{h})$ $(k\bar{l}h)$ $(k\bar{l}\bar{h})$ $(h\bar{l}k)$ $(\bar{h}l\bar{k})$ $(h\bar{l}k)$ $(h\bar{l}\bar{k})$	
			{ Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron (for $ h < l $) <i>Cube</i> & <i>octahedron</i> & <i>rhomb-dodecahedron</i> (for $ x < z $)	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}h\bar{l})$ $(h\bar{h}\bar{l})$ $(hh\bar{l})$ $(\bar{h}\bar{h}l)$ $(h\bar{h}l)$ $(\bar{h}hl)$ (lhh) $(l\bar{h}\bar{h})$ $(l\bar{h}h)$ $(l\bar{h}\bar{h})$ (lhh) $(l\bar{h}\bar{h})$ $(lh\bar{h})$ $(l\bar{h}h)$ (hlh) $(\bar{h}l\bar{h})$ $(h\bar{l}h)$ $(h\bar{l}\bar{h})$ $(h\bar{l}h)$ $(\bar{h}l\bar{h})$ $(h\bar{l}h)$ $(h\bar{l}\bar{h})$	
			Trigon-trioctahedron or trisoctahedron (for $ h > l $) <i>Cube truncated by octahedron</i> (for $ x > z $)		
			Tetrahexahedron or tetrakisshexahedron <i>Octahedron truncated by cube</i>	$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$ $(k0\bar{l})$ $(\bar{k}0\bar{l})$ $(k0l)$ $(\bar{k}0l)$ $(l0k)$ $(l0\bar{k})$ $(l0k)$ $(l0\bar{k})$ $(\bar{l}k0)$ $(\bar{l}\bar{k}0)$ $(lk0)$ $(l\bar{k}0)$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(0\bar{l}k)$ $(0l\bar{k})$ $(0lk)$ $(0l\bar{k})$	
12	<i>c</i>	.2	Rhomb-dodecahedron <i>Cuboctahedron</i> (<i>i</i>)	(011) $(0\bar{1}\bar{1})$ $(01\bar{1})$ $(0\bar{1}1)$ (101) $(10\bar{1})$ $(\bar{1}01)$ $(\bar{1}0\bar{1})$ (110) $(\bar{1}\bar{1}0)$ $(1\bar{1}0)$ $(\bar{1}10)$	
8	<i>b</i>	.3	Octahedron <i>Cube</i> (<i>g</i>)	(111) $(\bar{1}\bar{1}\bar{1})$ $(\bar{1}\bar{1}1)$ $(1\bar{1}\bar{1})$ $(\bar{1}\bar{1}1)$ $(11\bar{1})$ $(1\bar{1}1)$ $(\bar{1}1\bar{1})$	
6	<i>a</i>	4.	Cube or hexahedron <i>Octahedron</i> (<i>e</i>)	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$ (001) $(00\bar{1})$	
Symmetry of special projections					
			Along $[001]$	Along $[111]$	Along $[110]$
			$4mm$	$3m$	$2mm$

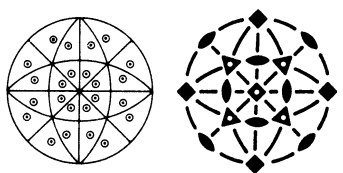
10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.2.2. *The 32 three-dimensional crystallographic point groups (cont.)*

CUBIC SYSTEM (cont.)							
$\bar{4}3m$	T_d						
24	d	1	Hexatetrahedron or hexakistetrahedron <i>Cube truncated by two tetrahedra (j)</i> Tetrahexahedron or tetrakisshexahedron <i>Octahedron truncated by cube</i>	(hkl) $(\bar{h}\bar{k}l)$ $(\bar{h}k\bar{l})$ $(h\bar{k}\bar{l})$ (lhk) $(\bar{l}\bar{h}\bar{k})$ $(\bar{l}hk)$ $(l\bar{h}k)$ (klh) $(\bar{k}\bar{l}h)$ $(\bar{k}lh)$ $(k\bar{l}h)$	(khl) $(\bar{k}\bar{h}l)$ $(k\bar{h}\bar{l})$ $(\bar{k}h\bar{l})$ (lkh) $(\bar{l}\bar{k}h)$ $(l\bar{k}\bar{h})$ $(\bar{l}k\bar{h})$ (hlk) $(\bar{h}\bar{l}k)$ $(h\bar{l}k)$ $(\bar{h}lk)$		
12	c	$.m$	{ Trigon-tritetrahedron or tristetrahedron (for $ h < l $) <i>Tetrahedron truncated by tetrahedron (i)</i> (for $ x < z $) Tetragon-tritetrahedron or deltohedron or delto-dodecahedron (for $ h > l $) <i>Cube & two tetrahedra (i)</i> (for $ x > z $) }	(hhl) $(\bar{h}\bar{h}l)$ $(\bar{h}h\bar{l})$ $(h\bar{h}\bar{l})$ (lhh) $(\bar{l}\bar{h}h)$ $(\bar{l}hh)$ $(l\bar{h}\bar{h})$ (hll) $(\bar{h}\bar{l}h)$ $(\bar{h}ll)$ $(h\bar{l}\bar{h})$			
			Rhomb-dodecahedron <i>Cuboctahedron</i>	(110) $(\bar{1}\bar{1}0)$ $(\bar{1}10)$ $(1\bar{1}0)$ (011) $(0\bar{1}\bar{1})$ $(0\bar{1}1)$ $(01\bar{1})$ (101) $(\bar{1}0\bar{1})$ $(10\bar{1})$ $(\bar{1}01)$			
6	b	$2.mm$	Cube or hexahedron <i>Octahedron (f)</i>	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$ (001) $(00\bar{1})$			
4	a	$.3m$	Tetrahedron <i>Tetrahedron (e)</i>	(111) $(\bar{1}\bar{1}\bar{1})$ $(\bar{1}\bar{1}1)$ $(1\bar{1}\bar{1})$ or $(\bar{1}\bar{1}\bar{1})$ $(1\bar{1}\bar{1})$ $(\bar{1}\bar{1}1)$ $(1\bar{1}\bar{1})$			
			Symmetry of special projections Along $[001]$ Along $[111]$ Along $[110]$ $4mm$ $3m$ m				

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.2. The 32 three-dimensional crystallographic point groups (cont.)

CUBIC SYSTEM (cont.)												
$m\bar{3}m$ $\frac{4}{m}\frac{3}{m}\frac{2}{m}$	O_h											
48	f	l	Hexaoctahedron or hexakisoctahedron Cube truncated by octahedron and by rhomb- dodecahedron (n)	(hkl) $(\bar{h}kl)$ $(h\bar{k}l)$ $(\bar{h}\bar{k}l)$ (khl) $(\bar{k}hl)$ $(kh\bar{l})$ $(\bar{k}\bar{h}\bar{l})$ (lhk) $(\bar{l}hk)$ $(l\bar{h}k)$ $(\bar{l}\bar{h}k)$ (lkh) $(\bar{l}k\bar{h})$ $(l\bar{h}k)$ $(\bar{l}k\bar{h})$ (klh) $(\bar{k}lh)$ $(k\bar{l}h)$ $(\bar{k}\bar{l}h)$ $(\bar{h}lk)$ $(\bar{h}lk)$ $(\bar{h}lk)$ $(\bar{h}lk)$ $(\bar{h}kl)$ (hkl) $(\bar{h}kl)$ (hkl) $(\bar{k}hl)$ (khl) $(\bar{k}hl)$ (khl) $(\bar{l}hk)$ $(\bar{l}hk)$ $(l\bar{h}k)$ $(\bar{l}hk)$ $(\bar{l}k\bar{h})$ (lkh) $(\bar{l}k\bar{h})$ $(\bar{l}k\bar{h})$ $(\bar{k}lh)$ $(k\bar{l}h)$ $(\bar{k}lh)$ $(\bar{k}lh)$ $(\bar{h}lk)$ $(h\bar{l}k)$ $(\bar{h}lk)$ $(\bar{h}lk)$								
24	e	$..m$	<table border="0"> <tr> <td rowspan="2">Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron (for $h < l$) Cube & octahedron & rhomb- dodecahedron (m) (for $x < z$)</td> <td rowspan="2">}</td> <td>(hhl) $(\bar{h}hl)$ $(\bar{h}hl)$ (hhl) $(hh\bar{l})$ $(\bar{h}h\bar{l})$ $(hh\bar{l})$ $(\bar{h}h\bar{l})$</td> </tr> <tr> <td>(lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$</td> </tr> <tr> <td rowspan="2">Trigon-trioctahedron or trisoctahedron (for $h > l$) Cube truncated by octahedron (m) (for $x > z$)</td> <td rowspan="2">}</td> <td>(hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$</td> </tr> <tr> <td></td> </tr> </table>	Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron (for $ h < l $) Cube & octahedron & rhomb- dodecahedron (m) (for $ x < z $)	}	(hhl) $(\bar{h}hl)$ $(\bar{h}hl)$ (hhl) $(hh\bar{l})$ $(\bar{h}h\bar{l})$ $(hh\bar{l})$ $(\bar{h}h\bar{l})$	(lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$	Trigon-trioctahedron or trisoctahedron (for $ h > l $) Cube truncated by octahedron (m) (for $ x > z $)	}	(hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$		(hhl) $(\bar{h}hl)$ $(\bar{h}hl)$ (hhl) $(hh\bar{l})$ $(\bar{h}h\bar{l})$ $(hh\bar{l})$ $(\bar{h}h\bar{l})$ (lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$ (hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$
Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron (for $ h < l $) Cube & octahedron & rhomb- dodecahedron (m) (for $ x < z $)	}	(hhl) $(\bar{h}hl)$ $(\bar{h}hl)$ (hhl) $(hh\bar{l})$ $(\bar{h}h\bar{l})$ $(hh\bar{l})$ $(\bar{h}h\bar{l})$										
		(lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$ $(\bar{l}hh)$ (lhh) $(\bar{l}hh)$										
Trigon-trioctahedron or trisoctahedron (for $ h > l $) Cube truncated by octahedron (m) (for $ x > z $)	}	(hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$ $(\bar{h}lh)$ (hlh) $(\bar{h}lh)$										
24	d	$m..$	Tetrahexahedron or tetrakisohedron Octahedron truncated by cube (k)	$(0kl)$ $(0\bar{k}l)$ $(0k\bar{l})$ $(0\bar{k}\bar{l})$ $(k0\bar{l})$ $(\bar{k}0\bar{l})$ $(k0\bar{l})$ $(\bar{k}0\bar{l})$ $(l0k)$ $(\bar{l}0k)$ $(\bar{l}0k)$ $(l0\bar{k})$ $(\bar{l}k0)$ $(\bar{l}k0)$ $(l0k)$ $(\bar{l}k0)$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(k\bar{l}0)$ $(\bar{k}l0)$ $(0\bar{l}k)$ $(0\bar{l}k)$ $(0\bar{l}k)$ $(0\bar{l}k)$								
12	c	$m.m2$	Rhomb-dodecahedron Cuboctahedron (i)	(011) $(0\bar{1}1)$ $(01\bar{1})$ $(0\bar{1}\bar{1})$ (101) $(\bar{1}01)$ $(10\bar{1})$ $(\bar{1}0\bar{1})$ (110) $(\bar{1}10)$ $(1\bar{1}0)$ $(\bar{1}\bar{1}0)$								
8	b	$.3m$	Octahedron Cube (g)	(111) $(\bar{1}\bar{1}1)$ $(\bar{1}1\bar{1})$ $(1\bar{1}\bar{1})$ $(1\bar{1}\bar{1})$ $(\bar{1}\bar{1}\bar{1})$ $(1\bar{1}1)$ $(\bar{1}11)$								
6	a	$4m.m$	Cube or hexahedron Octahedron (e)	(100) $(\bar{1}00)$ (010) $(0\bar{1}0)$ (001) $(00\bar{1})$								
Symmetry of special projections												
			Along $[001]$ $4mm$	Along $[111]$ $6mm$								
				Along $[110]$ $2mm$								

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

 Table 10.1.2.3. *The 47 crystallographic face and point forms, their names, eigensymmetries, and their occurrence in the crystallographic point groups (generating point groups)*

The oriented face (site) symmetries of the forms are given in parentheses after the Hermann–Mauguin symbol (column 6); a symbol such as $mm2(.m., m..)$ indicates that the form occurs in point group $mm2$ twice, with face (site) symmetries $.m.$ and $m..$. Basic (general and special) forms are printed in bold face, limiting (general and special) forms in normal type. The various settings of point groups 32 , $3m$, $\bar{3}m$, $\bar{4}2m$ and $6m2$ are connected by braces.

No.	Crystal form	Point form	Number of faces or points	Eigensymmetry	Generating point groups with oriented face (site) symmetries between parentheses
1	Pedion or monohedron	Single point	1	∞m	1(1); 2(2); $m(m)$; 3(3); 4(4); 6(6); $mm2(mm2)$; $4mm(4mm)$; $3m(3m)$; $6mm(6mm)$
2	Pinacoid or parallelohedron	Line segment through origin	2	$\frac{\infty}{m}m$	$\bar{1}(1)$; $2(1)$; $m(1)$; $\frac{2}{m}(2.m)$; 222(2... .2.. .2) ; $mm2(.m., m..)$; $mmm(2mm, m2m, mm2)$; $\bar{4}(2..)$; $\frac{4}{m}(4..)$; $422(4..)$, $\left\{ \begin{array}{l} \bar{4}2m(2.mm) \\ \bar{4}m2(2mm.) \end{array} \right.$; $\frac{4}{m}mm(4mm)$; $\bar{3}(3..)$; $\left\{ \begin{array}{l} 321(3..) \\ 312(3..) \\ 32(3.) \end{array} \right.$; $\left\{ \begin{array}{l} \bar{3}m1(3m.) \\ \bar{3}1m(3.m) \\ \bar{3}m1(3m) \end{array} \right.$; $\bar{6}(3..)$; $\frac{6}{m}(6..)$; 622(6..) ; $\left\{ \begin{array}{l} \bar{6}m2(3m.) \\ \bar{6}2m(3.m) \end{array} \right.$; $\frac{6}{m}mm(6mm)$
3	Sphenoid, dome, or dihedron	Line segment	2	$mm2$	2(1); $m(1)$; $mm2(.m., m..)$
4	Rhombic disphenoid or rhombic tetrahedron	Rhombic tetrahedron	4	222	222(1)
5	Rhombic pyramid	Rectangle	4	$mm2$	$mm2(1)$
6	Rhombic prism	Rectangle through origin	4	mmm	2/$m(1)$; 222(1)*; $mm2(1)$; $mmm(m.., m., .m)$
7	Rhombic dipyramid	Quad	8	mmm	$mmm(1)$
8	Tetragonal pyramid	Square	4	$4mm$	4(1); $4mm(..m., .m.)$
9	Tetragonal disphenoid or tetragonal tetrahedron	Tetragonal tetrahedron	4	$\bar{4}2m$	4(1) ; $\left\{ \begin{array}{l} \bar{4}2m(..m) \\ \bar{4}m2(.m.) \end{array} \right.$
10	Tetragonal prism	Square through origin	4	$\frac{4}{m}mm$	$4(1)$; $\bar{4}(1)$; $\frac{4}{m}(m..)$; 422(..2, .2.) ; $4mm(..m., .m.)$; $\left\{ \begin{array}{l} \bar{4}2m(.2.) \ \& \ \bar{4}2m(..m) \\ \bar{4}m2(..2) \ \& \ \bar{4}m2(.m.) \end{array} \right.$; $\frac{4}{m}mm(m.m2, m2m.)$
11	Tetragonal trapezohedron	Twisted tetragonal antiprism	8	422	422(1)
12	Ditetragonal pyramid	Truncated square	8	$4mm$	$4mm(1)$
13	Tetragonal scalenohedron	Tetragonal tetrahedron cut off by pinacoid	8	$\bar{4}2m$	$\left\{ \begin{array}{l} \bar{4}2m(1) \\ \bar{4}m2(1) \end{array} \right.$
14	Tetragonal dipyramid	Tetragonal prism	8	$\frac{4}{m}mm$	$\frac{4}{m}(1)$; $422(1)*$; $\left\{ \begin{array}{l} \bar{4}2m(1) \\ \bar{4}m2(1) \end{array} \right.$; $\frac{4}{m}mm(m., m.)$
15	Ditetragonal prism	Truncated square through origin	8	$\frac{4}{m}mm$	$422(1)$; $4mm(1)$; $\left\{ \begin{array}{l} \bar{4}2m(1) \\ \bar{4}m2(1) \end{array} \right.$; $\frac{4}{m}mm(m..)$
16	Ditetragonal dipyramid	Edge-truncated tetragonal prism	16	$\frac{4}{m}mm$	$\frac{4}{m}mm(1)$

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.3. *The 47 crystallographic face and point forms, their names, eigensymmetries, and their occurrence in the crystallographic point groups (generating point groups) (cont.)*

No.	Crystal form	Point form	Number of faces or points	Eigensymmetry	Generating point groups with oriented face (site) symmetries between parentheses
17	Trigonal pyramid	Trigon	3	$3m$	$3(1); \left\{ \begin{array}{l} 3m1(.m.) \\ 31m(..m) \\ 3m(.m) \end{array} \right.$
18	Trigonal prism	Trigon through origin	3	$\bar{6}2m$	$3(1); \left\{ \begin{array}{l} 321(.2.) \\ 312(..2) \\ 32(.2) \end{array} \right. \left\{ \begin{array}{l} 3m1(.m.) \\ 31m(..m) \\ 3m(.m) \end{array} \right.;$ $\bar{6}(m..); \left\{ \begin{array}{l} \bar{6}m2(mm2) \\ \bar{6}2m(m2m) \end{array} \right.$
19	Trigonal trapezohedron	Twisted trigonal antiprism	6	32	$\left\{ \begin{array}{l} 321(1) \\ 312(1) \\ 32(1) \end{array} \right.$
20	Ditrigonal pyramid	Truncated trigon	6	$3m$	$3m(1)$
21	Rhombohedron	Trigonal antiprism	6	$\bar{3}m$	$\bar{3}(1); \left\{ \begin{array}{l} 321(1) \\ 312(1) \\ 32(1) \end{array} \right. \left\{ \begin{array}{l} \bar{3}m1(.m.) \\ \bar{3}1m(..m) \\ \bar{3}m(.m) \end{array} \right.$
22	Ditrigonal prism	Truncated trigon through origin	6	$\bar{6}2m$	$\left\{ \begin{array}{l} 321(1) \\ 312(1) \\ 32(1) \end{array} \right. \left\{ \begin{array}{l} 3m1(1) \\ 31m(1) \\ 3m(1) \end{array} \right.;$ $\left\{ \begin{array}{l} \bar{6}m2(m..) \\ \bar{6}2m(m..) \end{array} \right.$
23	Hexagonal pyramid	Hexagon	6	$6mm$	$\left\{ \begin{array}{l} 3m1(1) \\ 31m(1); 6(1); 6mm(..m..m.) \\ 3m(1) \end{array} \right.$
24	Trigonal dipyrmaid	Trigonal prism	6	$\bar{6}2m$	$\left\{ \begin{array}{l} 321(1) \\ 312(1); \bar{6}(1); \\ 32(1) \end{array} \right. \left\{ \begin{array}{l} \bar{6}m2(.m.) \\ \bar{6}2m(..m) \end{array} \right.$
25	Hexagonal prism	Hexagon through origin	6	$\frac{6}{m}mm$	$\bar{3}(1); \left\{ \begin{array}{l} 321(1) \\ 312(1); \\ 32(1) \end{array} \right. \left\{ \begin{array}{l} 3m1(1) \\ 31m(1) \\ 3m(1) \end{array} \right.;$ $\left\{ \begin{array}{l} \bar{3}m1(.2.) \& \bar{3}m1(.m.) \\ \bar{3}1m(..2) \& \bar{3}1m(..m); \\ \bar{3}m(.2) \& \bar{3}m(.m) \end{array} \right.;$ $6(1); \frac{6}{m}(m..); 622(.2., .2.);$ $6mm(..m, .m.); \left\{ \begin{array}{l} \bar{6}m2(m..) \\ \bar{6}2m(m..) \end{array} \right.;$ $\frac{6}{m}mm(m2m, mm2)$
26	Ditrigonal scalenohedron or hexagonal scalenohedron	Trigonal antiprism sliced off by pinacoid	12	$\bar{3}m$	$\left\{ \begin{array}{l} \bar{3}m1(1) \\ \bar{3}1m(1) \\ \bar{3}m(1) \end{array} \right.$
27	Hexagonal trapezohedron	Twisted hexagonal antiprism	12	622	$622(1)$
28	Dihexagonal pyramid	Truncated hexagon	12	$6mm$	$6mm(1)$
29	Ditrigonal dipyrmaid	Edge-truncated trigonal prism	12	$\bar{6}2m$	$\left\{ \begin{array}{l} \bar{6}m2(1) \\ \bar{6}2m(1) \end{array} \right.$
30	Dihexagonal prism	Truncated hexagon	12	$\frac{6}{m}mm$	$\left\{ \begin{array}{l} \bar{3}m1(1) \\ \bar{3}1m(1); 622(1); 6mm(1); \\ \bar{3}m(1) \end{array} \right.;$ $\frac{6}{m}mm(m..)$

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

 Table 10.1.2.3. *The 47 crystallographic face and point forms, their names, eigensymmetries, and their occurrence in the crystallographic point groups (generating point groups) (cont.)*

No.	Crystal form	Point form	Number of faces or points	Eigensymmetry	Generating point groups with oriented face (site) symmetries between parentheses
31	Hexagonal dipyramid	Hexagonal prism	12	$\frac{6}{m}mm$	$\left\{ \begin{array}{l} \bar{3}m1(1) \\ \bar{3}1m(1); \frac{6}{m}(1); 622(1)^*; \\ \bar{3}m(1) \end{array} \right.$ $\left\{ \begin{array}{l} \bar{6}m2(1); \frac{6}{m}mm(.., .m.) \\ \bar{6}2m(1) \end{array} \right.$
32	Dihexagonal dipyramid	Edge-truncated hexagonal prism	24	$\frac{6}{m}mm$	$\frac{6}{m}mm(1)$
33	Tetrahedron	Tetrahedron	4	$\bar{4}3m$	$23(.3); \bar{4}3m(.3m)$
34	Cube or hexahedron	Octahedron	6	$m\bar{3}m$	$23(2.); m\bar{3}(2mm.); 432(4.); \bar{4}3m(2.mm); m\bar{3}m(4m.m)$
35	Octahedron	Cube	8	$m\bar{3}m$	$m\bar{3}(.3); 432(.3); m\bar{3}m(.3m)$
36	Pentagon-tritetrahedron or tetartoid or tetrahedral pentagon-dodecahedron	Snub tetrahedron (=pentagon-tritetrahedron + two tetrahedra)	12	23	$23(1)$
37	Pentagon-dodecahedron or dihexahedron or pyritohedron	Irregular icosahedron (= pentagon-dodecahedron + octahedron)	12	$m\bar{3}$	$23(1); m\bar{3}(m.)$
38	Tetragon-tritetrahedron or deltohedron or deltoid-dodecahedron	Cube and two tetrahedra	12	$\bar{4}3m$	$23(1); \bar{4}3m(..m)$
39	Trigon-tritetrahedron or tristetrahedron	Tetrahedron truncated by tetrahedron	12	$\bar{4}3m$	$23(1); \bar{4}3m(..m)$
40	Rhomb-dodecahedron	Cuboctahedron	12	$m\bar{3}m$	$23(1); m\bar{3}(m.); 432(..2); 43m(..m); m\bar{3}m(m.m2)$
41	Didodecahedron or diploid or dyakisdodecahedron	Cube & octahedron & pentagon-dodecahedron	24	$m\bar{3}$	$m\bar{3}(1)$
42	Trigon-trioctahedron or trisoctahedron	Cube truncated by octahedron	24	$m\bar{3}m$	$m\bar{3}(1); 432(1); m\bar{3}m(..m)$
43	Tetragon-trioctahedron or trapezohedron or deltoid-icositetrahedron	Cube & octahedron & rhomb-dodecahedron	24	$m\bar{3}m$	$m\bar{3}(1); 432(1); m\bar{3}m(..m)$
44	Pentagon-trioctahedron or gyroid	Cube + octahedron + pentagon-trioctahedron	24	432	$432(1)$
45	Hexatetrahedron or hexakistetrahedron	Cube truncated by two tetrahedra	24	$\bar{4}3m$	$\bar{4}3m(1)$
46	Tetrahexahedron or tetrakisshexahedron	Octahedron truncated by cube	24	$m\bar{3}m$	$432(1); \bar{4}3m(1); m\bar{3}m(m.)$
47	Hexaoctahedron or hexakisoctahedron	Cube truncated by octahedron and by rhomb-dodecahedron	48	$m\bar{3}m$	$m\bar{3}m(1)$

* These limiting forms occur in three or two non-equivalent orientations (different types of limiting forms); cf. Table 10.1.2.2.

† In point groups $\bar{4}2m$ and $\bar{3}m$, the tetragonal prism and the hexagonal prism occur twice, as a 'basic special form' and as a 'limiting special form'. In these cases, the point groups are listed twice, as $\bar{4}2m(.2)$ & $\bar{4}2m(..m)$ and as $\bar{3}m1(.2)$ & $\bar{3}m1(m.)$.

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.2.4. Names and symbols of the 32 crystal classes

System used in this volume	Point group		Schoenflies symbol	Class names	
	International symbol			Groth (1921)	Friedel (1926)
	Short	Full			
Triclinic	1 $\bar{1}$	1 $\bar{1}$	C_1 $C_i(S_2)$	Pedial (asymmetric) Pinacoidal	Hemihedry Holohedry
Monoclinic	2 m $2/m$	2 m $\frac{2}{m}$	C_2 $C_s(C_{1h})$ C_{2h}	Sphenoidal Domestic Prismatic	Holoaxial hemihedry Antihemihedry Holohedry
Orthorhombic	222 $mm2$ mmm	222 $mm2$ $\frac{2}{m} \frac{2}{m} \frac{2}{m}$	$D_2(V)$ C_{2v} $D_{2h}(V_h)$	Disphenoidal Pyramidal Dipyramidal	Holoaxial hemihedry Antihemihedry Holohedry
Tetragonal	4 $\bar{4}$ $4/m$ 422 $4mm$ $\bar{4}2m$ $4/mmm$	4 $\bar{4}$ $\frac{4}{m}$ 422 $4mm$ $\bar{4}2m$ $\frac{4}{m} \frac{2}{m} \frac{2}{m}$	C_4 S_4 C_{4h} D_4 C_{4v} $D_{2d}(V_d)$ D_{4h}	Pyramidal Disphenoidal Dipyramidal Trapezohedral Ditetragonal-pyramidal Scalenohedral Ditetragonal-dipyramidal	Tetartohedry with 4-axis Sphenohedral tetartohedry Parahemihedry Holoaxial hemihedry Antihemihedry with 4-axis Sphenohedral antihemihedry Holohedry
Trigonal	3 $\bar{3}$ 32 $3m$ $\bar{3}m$	3 $\bar{3}$ 32 $3m$ $\frac{3}{2} \frac{2}{m}$	C_3 $C_{3i}(S_6)$ D_3 C_{3v} D_{3d}	Pyramidal Rhombohedral Trapezohedral Ditrigonal-pyramidal Ditrigonal-scalenohedral	<i>Hexagonal</i> Ogdohedry Paratetartohedry Holoaxial tetartohedry with 3-axis Hemimorphic antitetartohedry Parahemihedry with 3-axis <i>Rhombohedral</i> Tetartohedry Parahemihedry Holoaxial hemihedry Antihemihedry Holohedry
Hexagonal	6 $\bar{6}$ $6/m$ 622 $6mm$ $\bar{6}2m$ $6/mmm$	6 $\bar{6}$ $\frac{6}{m}$ 622 $6mm$ $\bar{6}2m$ $\frac{6}{m} \frac{2}{m} \frac{2}{m}$	C_6 C_{3h} C_{6h} D_6 C_{6v} D_{3h} D_{6h}	Pyramidal Trigonal-dipyramidal Dipyramidal Trapezohedral Dihexagonal-pyramidal Ditrigonal-dipyramidal Dihexagonal-dipyramidal	Tetartohedry with 6-axis Trigonal antitetartohedry Parahemihedry with 6-axis Holoaxial hemihedry Antihemihedry with 6-axis Trigonal antihemihedry Holohedry
Cubic	23 $m\bar{3}$ 432 $\bar{4}3m$ $m\bar{3}m$	23 $\frac{2}{3} \frac{3}{m}$ 432 $\bar{4}3m$ $\frac{4}{3} \frac{2}{m} \frac{2}{m}$	T T_h O T_d O_h	Tetrahedral-pentagondodecahedral (= tetartoidal) Disdodecahedral (= diploidal) Pentagon-icositetrahedral (= gyroidal) Hexakistetrahedral (= hextetrahedral) Hexakisoctahedral (= hexooctahedral)	Tetartohedry Parahemihedry Holoaxial hemihedry Antihemihedry Holohedry

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

(2) In point group 32, the general form is a trigonal trapezohedron $\{hkl\}$; this form can be considered as two opposite trigonal pyramids, rotated with respect to each other by an angle χ . The trapezohedron changes into the limiting forms 'trigonal dipyramid' $\{hhl\}$ for $\chi = 0^\circ$ and 'rhombohedral' $\{h0l\}$ for $\chi = 60^\circ$.

(vii) One and the same type of polyhedron can occur as a general, special or limiting form.

Examples

(1) A tetragonal dipyramid is a general form in point group $4/m$, a special form in point group $4/mmm$ and a limiting general form in point groups 422 and $42m$.

(2) A tetragonal prism appears in point group $\bar{4}2m$ both as a basic special form (4b) and as a limiting special form (4c).

(viii) A peculiarity occurs for the cubic point groups. Here the crystal forms $\{hhl\}$ are realized as two topologically different kinds of polyhedra with the same face symmetry, multiplicity and, in addition, the same *eigensymmetry*. The realization of one or other of these forms depends upon whether the Miller indices obey the conditions $|h| > |l|$ or $|h| < |l|$, i.e. whether, in the stereographic projection, a face pole is located between the directions [110] and [111] or between the directions [111] and [001]. These two kinds of polyhedra have to be considered as two *realizations of one type* of crystal form because their face poles are located on the same set of conjugate symmetry elements. Similar considerations apply to the point forms x, x, z .

In the point groups $m\bar{3}m$ and $\bar{4}3m$, the two kinds of polyhedra represent two realizations of one *special* 'Wyckoff position'; hence, they have the same Wyckoff letter. In the groups 23, $m\bar{3}$ and 432, they represent two realizations of the same type of limiting *general* forms. In the tables of the cubic point groups, the two entries are always connected by braces.

The same kind of peculiarity occurs for the two icosahedral point groups, as mentioned in Section 10.1.4 and listed in Table 10.1.4.3.

10.1.2.5. Names and symbols of the crystal classes

Several different sets of names have been devised for the 32 crystal classes. Their use, however, has greatly declined since the introduction of the international point-group symbols. As examples, two sets (both translated into English) that are frequently found in the literature are given in Table 10.1.2.4. To the name of the class the name of the system has to be added: e.g. 'tetragonal pyramidal' or 'tetragonal tetartohedry'.

Note that Friedel (1926) based his nomenclature on the point symmetry of the lattice. Hence, two names are given for the five trigonal point groups, depending whether the lattice is hexagonal or rhombohedral: e.g. 'hexagonal ogdohedry' and 'rhombohedral tetartohedry'.

10.1.3. Subgroups and supergroups of the crystallographic point groups

In this section, the sub- and supergroup relations between the crystallographic point groups are presented in the form of a 'family tree'.* Figs. 10.1.3.1 and 10.1.3.2 apply to two and three dimensions. The sub- and supergroup relations between two groups are represented by solid or dashed lines. For a given point group \mathcal{P} of order k_P the lines to groups of lower order connect \mathcal{P} with all its *maximal subgroups* \mathcal{H} with orders k_H ; the index $[i]$ of each subgroup is given by the ratio of the orders k_P/k_H . The lines to groups of higher order connect \mathcal{P} with all its *minimal supergroups* \mathcal{S}

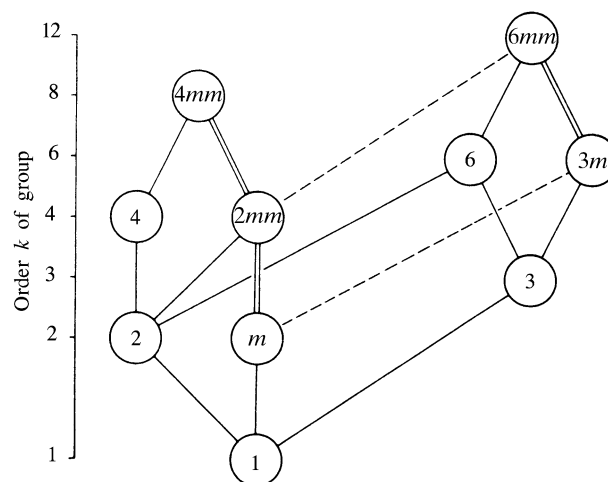


Fig. 10.1.3.1. Maximal subgroups and minimal supergroups of the two-dimensional crystallographic point groups. Solid lines indicate maximal normal subgroups; double solid lines mean that there are two maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. The group orders are given on the left.

with orders k_S ; the index $[i]$ of each supergroup is given by the ratio k_S/k_P . In other words: if the diagram is read downwards, subgroup relations are displayed; if it is read upwards, supergroup relations are revealed. The index is always an integer (theorem of Lagrange) and can be easily obtained from the group orders given on the left of the diagrams. The highest index of a maximal subgroup is [3] for two dimensions and [4] for three dimensions.

Two important kinds of subgroups, namely sets of conjugate subgroups and normal subgroups, are distinguished by dashed and solid lines. They are characterized as follows:

The subgroups $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ of a group \mathcal{P} are *conjugate* subgroups if $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ are symmetrically equivalent in \mathcal{P} , i.e. if for every pair $\mathcal{H}_i, \mathcal{H}_j$ at least one symmetry operation W of \mathcal{P} exists which maps \mathcal{H}_i onto \mathcal{H}_j : $W^{-1}\mathcal{H}_iW = \mathcal{H}_j$; cf. Section 8.3.6.

Examples

(1) Point group $3m$ has three different mirror planes which are equivalent due to the threefold axis. In each of the three maximal subgroups of type m , one of these mirror planes is retained. Hence, the three subgroups m are conjugate in $3m$. This set of conjugate subgroups is represented by *one* dashed line in Figs. 10.1.3.1 and 10.1.3.2.

(2) Similarly, group 432 has three maximal conjugate subgroups of type 422 and four maximal conjugate subgroups of type 32.

The subgroup \mathcal{H} of a group \mathcal{P} is a *normal* (or invariant) subgroup if *no* subgroup \mathcal{H}' of \mathcal{P} exists that is conjugate to \mathcal{H} in \mathcal{P} . Note that this does not imply that \mathcal{H} is also a normal subgroup of any supergroup of \mathcal{P} . Subgroups of index [2] are always normal and maximal. (The role of normal subgroups for the structure of space groups is discussed in Section 8.1.6.)

Examples

(1) Fig. 10.1.3.2 shows two solid lines between point groups 422 and 222, indicating that 422 has two maximal normal subgroups 222 of index [2]. The symmetry elements of one subgroup are rotated by 45° around the c axis with respect to those of the other subgroup. Thus, in one subgroup the symmetry elements of the two secondary, in the other those of the two tertiary tetragonal symmetry directions (cf. Table 2.2.4.1) are retained, whereas the primary twofold axis is the same for both subgroups. There exists no symmetry operation of 422 that maps one subgroup onto the other. This is illustrated by the stereograms below. The two normal subgroups can be indicated by the 'oriented

* This type of diagram was first used in *IT*(1935); in *IT*(1952) a somewhat different approach was employed.

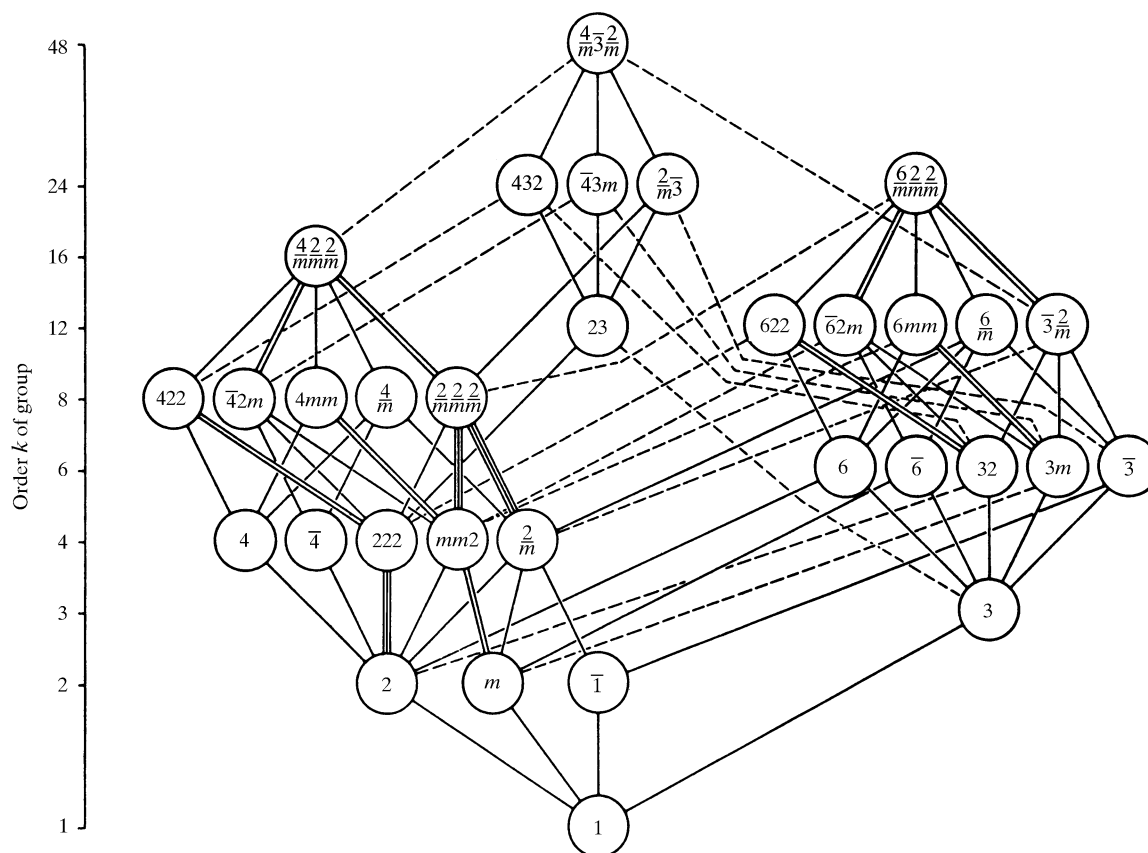
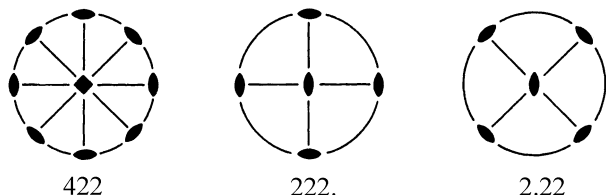


Fig. 10.1.3.2. Maximal subgroups and minimal supergroups of the three-dimensional crystallographic point groups. Solid lines indicate maximal normal subgroups; double or triple solid lines mean that there are two or three maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. The group orders are given on the left. Full Hermann-Mauguin symbols are used.

symbols' 222. and 2.22.



(2) Similarly, group 432 has one maximal normal subgroup, 23.

Figs. 10.1.3.1 and 10.1.3.2 show that there exist two 'summits' in both two and three dimensions from which all other point groups can be derived by 'chains' of maximal subgroups. These summits are formed by the square and the hexagonal holohedry in two dimensions and by the cubic and the hexagonal holohedry in three dimensions.

The sub- and supergroups of the point groups are useful both in their own right and as basis of the *translationengleiche* or *t* subgroups and supergroups of space groups; this is set out in Section 2.2.15. Tables of the sub- and supergroups of the plane groups and space groups are contained in Parts 6 and 7. A general discussion of sub- and supergroups of crystallographic groups, together with further explanations and examples, is given in Section 8.3.3.

10.1.4. Noncrystallographic point groups

10.1.4.1. Description of general point groups

In Sections 10.1.2 and 10.1.3, only the 32 *crystallographic* point groups (crystal classes) are considered. In addition, infinitely many *noncrystallographic* point groups exist that are of interest as possible symmetries of molecules and of quasicrystals and as

approximate local site symmetries in crystals. Crystallographic and noncrystallographic point groups are collected here under the name *general point groups*. They are reviewed in this section and listed in Tables 10.1.4.1 to 10.1.4.3.

Because of the infinite number of these groups only *classes of general point groups* (*general classes*)* can be listed. They are grouped into *general systems*, which are similar to the crystal systems. The 'general classes' are of two kinds: in the cubic, icosahedral, circular, cylindrical and spherical system, each general class contains *one* point group only, whereas in the $4N$ -gonal, $(2N + 1)$ -gonal and $(4N + 2)$ -gonal system, each general class contains *infinitely* many point groups, which differ in their principal n -fold symmetry axis, with $n = 4, 8, 12, \dots$ for the $4N$ -gonal system, $n = 1, 3, 5, \dots$ for the $(2N + 1)$ -gonal system and $n = 2, 6, 10, \dots$ for the $(4N + 2)$ -gonal system.

Furthermore, some general point groups are of order infinity because they contain symmetry axes (rotation or rotoinversion axes) of order infinity† (∞ -fold axes). These point groups occur in the

* The 'classes of general point groups' are not the same as the commonly used 'crystal classes' because some of them contain point groups of *different orders*. All these orders, however, follow a common scheme. In this sense, the 'general classes' are an extension of the concept of (geometric) crystal classes. For example, the general class mmm of the $4N$ -gonal system contains the point groups $4mm$ (tetragonal), $8mm$ (octagonal), $12mm$ (dodecagonal), $16mm$ etc.

† The axes of order infinity, as considered here, do not correspond to cyclic groups (as do the axes of finite order) because there is no smallest rotation from which all other rotations can be derived as higher powers, *i.e.* by successive application. Instead, rotations of all possible angles exist. Nevertheless, it is customary to symbolize these axes as ∞ or C_∞ ; note that the Hermann-Mauguin symbols ∞/m and ∞ are equivalent, and so are the Schoenflies symbols $C_{\infty h}$, S_∞ and $C_{\infty i}$. (There exist also axes of order infinity that do correspond to cyclic groups, namely axes based upon smallest rotations with irrational values of the rotation angle.)

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

circular system (two dimensions) and in the cylindrical and spherical systems (three dimensions).

The Hermann–Mauguin and Schoenflies symbols for the general point groups follow the rules of the crystallographic point groups (cf. Section 2.2.4 and Chapter 12.1). This extends also to the infinite groups where symbols like ∞m or $C_{\infty v}$ are immediately obvious.

In *two dimensions* (Table 10.1.4.1), the eight general classes are collected into three systems. Two of these, the $4N$ -gonal and the $(4N + 2)$ -gonal systems, contain only point groups of finite order with one n -fold rotation point each. These systems are generalizations of the square and hexagonal crystal systems. The circular system consists of two infinite point groups, with one ∞ -fold rotation point each.

In *three dimensions* (Table 10.1.4.2), the 28 general classes are collected into seven systems. Three of these, the $4N$ -gonal, the $(2N + 1)$ -gonal and the $(4N + 2)$ -gonal systems,* contain only point groups of finite order with one principal n -fold symmetry axis each. These systems are generalizations of the tetragonal, trigonal, and hexagonal crystal systems (cf. Table 10.1.1.2). The five cubic groups are well known as crystallographic groups. The two icosahedral groups of orders 60 and 120, characterized by special combinations of twofold, threefold and fivefold symmetry axes, are discussed in more detail below. The groups of the cylindrical and the spherical systems are all of order infinity; they describe the symmetries of cylinders, cones, rotation ellipsoids, spheres *etc.*

It is possible to define the three-dimensional point groups on the basis of either rotoinversion axes \bar{n} or rotoreflection axes \tilde{n} . The equivalence between these two descriptions is apparent from the following examples:

$$\begin{array}{llll} n = 4N & : & \bar{4} = \tilde{4} & \bar{8} = \tilde{8} \quad \dots \quad \bar{n} = \tilde{n} \\ n = 2N + 1 & : & \bar{1} = \tilde{2} & \bar{3} = \tilde{6} = 3 \times \bar{1} \dots \quad \bar{n} = \tilde{2n} = n \times \bar{1} \\ n = 4N + 2 & : & \bar{2} = \tilde{1} = m & \bar{6} = \tilde{3} = 3/m \quad \dots \quad \bar{n} = \tilde{\frac{1}{2}n} = \frac{1}{2}n/m. \end{array}$$

In the present tables, the standard convention of using rotoinversion axes is followed.

Tables 10.1.4.1 and 10.1.4.2 contain for each class its general Hermann–Mauguin and Schoenflies symbols, the group order and the names of the general face form and its dual, the general point form.† Special and limiting forms are not given, nor are ‘Miller indices’ (hkl) and point coordinates x, y, z . They can be derived easily from Tables 10.1.2.1 and 10.1.2.2 for the crystallographic groups.‡

10.1.4.2. The two icosahedral groups

The two point groups 235 and $m\bar{3}5$ of the icosahedral system (orders 60 and 120) are of particular interest among the noncrystallographic groups because of the occurrence of fivefold axes and their increasing importance as symmetries of molecules (viruses), of quasicrystals, and as approximate local site symmetries in crystals (alloys, B_{12} icosahedron). Furthermore, they contain as special forms the two noncrystallographic *platonic solids*, the

regular icosahedron (20 faces, 12 vertices) and its dual, the regular pentagon-dodecahedron (12 faces, 20 vertices).

The icosahedral groups (cf. diagrams in Table 10.1.4.3) are characterized by six fivefold axes that include angles of 63.43° . Each fivefold axis is surrounded by five threefold and five twofold axes, with angular distances of 37.38° between a fivefold and a threefold axis and of 31.72° between a fivefold and a twofold axis. The angles between neighbouring threefold axes are 41.81° , between neighbouring twofold axes 36° . The smallest angle between a threefold and a twofold axis is 20.90° .

Each of the six fivefold axes is perpendicular to five twofold axes; there are thus six maximal conjugate pentagonal subgroups of types 52 (for 235) and $\bar{5}m$ (for $m\bar{3}5$) with index [6]. Each of the ten threefold axes is perpendicular to three twofold axes, leading to ten maximal conjugate trigonal subgroups of types 32 (for 235) and $\bar{3}m$ (for $m\bar{3}5$) with index [10]. There occur, furthermore, five maximal conjugate cubic subgroups of types 23 (for 235) and $m\bar{3}$ (for $m\bar{3}5$) with index [5].

The two icosahedral groups are listed in Table 10.1.4.3, in a form similar to the cubic point groups in Table 10.1.2.2. Each group is illustrated by stereographic projections of the symmetry elements and the general face poles (general points); the complete sets of symmetry elements are listed below the stereograms. Both groups are referred to a cubic coordinate system, with the coordinate axes along three twofold rotation axes and with four threefold axes along the body diagonals. This relation is well brought out by symbolizing these groups as 235 and $m\bar{3}5$ instead of the customary symbols 532 and $\bar{5}3m$.

The table contains also the multiplicities, the Wyckoff letters and the names of the general and special face forms and their duals, the point forms, as well as the oriented face- and site-symmetry symbols. In the icosahedral ‘holohedry’ $m\bar{3}5$, the *special* ‘Wyckoff position’ $60d$ occurs in three realizations, *i.e.* with three types of polyhedra. In 235, however, these three types of polyhedra are different realizations of the limiting *general* forms, which depend on the location of the poles with respect to the axes 2, 3 and 5. For this reason, the three entries are connected by braces; cf. Section 10.1.2.4, *Notes on crystal and point forms*, item (viii).

Not included are the sets of equivalent Miller indices and point coordinates. Instead, only the ‘initial’ triplets (hkl) and x, y, z for each type of form are listed. The complete sets of indices and coordinates can be obtained in two steps§ as follows:

(i) For the face forms the cubic point groups 23 and $m\bar{3}$ (Table 10.1.2.2), and for the point forms the cubic space groups $P23$ (195) and $Pm\bar{3}$ (200) have to be considered. For each ‘initial’ triplet (hkl), the set of Miller indices of the (general or special) crystal form with the same face symmetry in 23 (for group 235) or $m\bar{3}$ (for $m\bar{3}5$) is taken. For each ‘initial’ triplet x, y, z , the coordinate triplets of the (general or special) position with the same site symmetry in $P23$ or $Pm\bar{3}$ are taken; this procedure is similar to that described in Section 10.1.2.3 for the crystallographic point forms.

(ii) To obtain the complete set of icosahedral Miller indices and point coordinates, the ‘cubic’ (hkl) triplets (as rows) and x, y, z triplets (as columns) have to be multiplied with the identity matrix and with

(a) the matrices Y, Y^2, Y^3 and Y^4 for the Miller indices;

* Here, the $(2N + 1)$ -gonal and the $(4N + 2)$ -gonal systems are distinguished in order to bring out the analogy with the trigonal and the hexagonal crystal systems. They could equally well be combined into one, in correspondence with the hexagonal ‘crystal family’ (cf. Chapter 2.1).

† The noncrystallographic face and point forms are extensions of the corresponding crystallographic forms: cf. Section 10.1.2.4, *Notes on crystal and point forms*. The name *streptohedron* applies to the general face forms of point groups \bar{n} with $n = 4N$ and $n = 2N + 1$; it is thus a generalization of the tetragonal disphenoid or tetragonal tetrahedron (4) and the rhombohedron ($\bar{3}$).

‡ The term ‘Miller indices’ is used here also for the noncrystallographic point groups. Note that these indices do not have to be integers or rational numbers, as for the crystallographic point groups. Irrational indices, however, can always be closely approximated by integers, quite often even by small integers.

§ A one-step procedure applies to the icosahedral ‘Wyckoff position’ $12a$, the face poles and points of which are located on the fivefold axes. Here, step (ii) is redundant and can be omitted. The forms $\{01\tau\}$ and $0, y, \tau y$ are contained in the cubic point groups 23 and $m\bar{3}$ and in the cubic space groups $P23$ and $Pm\bar{3}$ as limiting cases of Wyckoff positions $\{0kl\}$ and $0, y, z$ with specialized (irrational) values of the indices and coordinates. In geometric terms, the regular pentagon-dodecahedron is a noncrystallographic ‘limiting polyhedron’ of the ‘crystallographic’ pentagon-dodecahedron and the regular icosahedron is a ‘limiting polyhedron’ of the ‘irregular’ icosahedron (cf. Section 10.1.2.2, *Crystal and point forms*).

10. POINT GROUPS AND CRYSTAL CLASSES

Table 10.1.4.1. Classes of general point groups in two dimensions ($N = \text{integer} \geq 0$)

General Hermann–Mauguin symbol	Order of group	General edge form	General point form	Crystallographic groups
$4N$ -gonal system (n -fold rotation point with $n = 4N$)				
n nmm	n $2n$	Regular n -gon Semiregular di- n -gon	Regular n -gon Truncated n -gon	4 $4mm$
$(4N + 2)$ -gonal system (n -fold or $\frac{1}{2}n$ -fold rotation point with $n = 4N + 2$)				
$\frac{1}{2}n$ $\frac{1}{2}nm$ n nmm	$\frac{1}{2}n$ n n $2n$	Regular $\frac{1}{2}n$ -gon Semiregular di- $\frac{1}{2}n$ -gon Regular n -gon Semiregular di- n -gon	Regular $\frac{1}{2}n$ -gon Truncated $\frac{1}{2}n$ -gon Regular n -gon Truncated n -gon	1, 3 $m, 3m$ 2, 6 $2mm, 6mm$
Circular system *				
∞ ∞m	∞ ∞	Rotating circle Stationary circle	Rotating circle Stationary circle	– –

* A rotating circle has no mirror lines; there exist two enantiomorphic circles with opposite senses of rotation. A stationary circle has infinitely many mirror lines through its centre.

Table 10.1.4.2. Classes of general point groups in three dimensions ($N = \text{integer} \geq 0$)

Short general Hermann–Mauguin symbol, followed by full symbol where different	Schoenflies symbol	Order of group	General face form	General point form	Crystallographic groups
$4N$ -gonal system (single n -fold symmetry axis with $n = 4N$)					
n \bar{n} n/m $n22$ nmm $\bar{n}2m$ $n/mmm, \frac{n\ 2\ 2}{m\ m\ m}$	C_n S_n C_{nh} D_n C_{nv} $D_{\frac{1}{2}nd}$ D_{nh}	n n $2n$ $2n$ $2n$ $2n$ $4n$	n -gonal pyramid $\frac{1}{2}n$ -gonal streptohedron n -gonal dipyramid n -gonal trapezohedron Di- n -gonal pyramid n -gonal scalenohedron Di- n -gonal dipyramid	Regular n -gon $\frac{1}{2}n$ -gonal antiprism n -gonal prism Twisted n -gonal antiprism Truncated n -gon $\frac{1}{2}n$ -gonal antiprism sliced off by pinacoid Edge-truncated n -gonal prism	4 $\bar{4}$ $4/m$ 422 $4mm$ $\bar{4}2m$ $4/mmm$
$(2N + 1)$ -gonal system (single n -fold symmetry axis with $n = 2N + 1$)					
n $\bar{n} = n \times \bar{1}$ $n2$ nm $\bar{n}m, \bar{n} \frac{2}{m}$	C_n C_{ni} D_n C_{nv} D_{nd}	n $2n$ $2n$ $2n$ $4n$	n -gonal pyramid n -gonal streptohedron n -gonal trapezohedron Di- n -gonal pyramid Di- n -gonal scalenohedron	Regular n -gon n -gonal antiprism Twisted n -gonal antiprism Truncated n -gon n -gonal antiprism sliced off by pinacoid	1, 3 $\bar{1}, \bar{3} = 3 \times \bar{1}$ 32 $3m$ $\bar{3}m$
$(4N + 2)$ -gonal system (single n -fold symmetry axis with $n = 4N + 2$)					
n $\bar{n} = \frac{1}{2}n/m$ n/m $n22$ nmm $\bar{n}2m = \frac{1}{2}n/m2m$ $n/mmm, \frac{n\ 2\ 2}{m\ m\ m}$	C_n $C_{\frac{1}{2}nh}$ C_{nh} D_n C_{nv} $D_{\frac{1}{2}nh}$ D_{nh}	n n $2n$ $2n$ $2n$ $2n$ $4n$	n -gonal pyramid $\frac{1}{2}n$ -gonal dipyramid n -gonal dipyramid n -gonal trapezohedron Di- n -gonal pyramid Di- $\frac{1}{2}n$ -gonal dipyramid Di- n -gonal dipyramid	Regular n -gon $\frac{1}{2}n$ -gonal prism n -gonal prism Twisted n -gonal antiprism Truncated n -gon Truncated $\frac{1}{2}n$ -gonal prism Edge-truncated n -gonal prism	2, 6 $\bar{2} \equiv m, \bar{6} \equiv 3/m$ $2/m, 6/m$ 222, 622 $mm2, 6mm$ $\bar{6}2m$ $mmm, 6/mmm$

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.4.2. *Classes of general point groups in three dimensions ($N = \text{integer} \geq 0$) (cont.)*

Short general Hermann–Mauguin symbol, followed by full symbol where different	Schoenflies symbol	Order of group	General face form	General point form	Crystallographic groups
Cubic system (for details see Table 10.1.2.2)					
23 $m\bar{3}, \frac{2}{m}\bar{3}$	T T_h	12 24	Pentagon-tritetrahedron Didodecahedron	Snub tetrahedron Cube & octahedron & pentagon-dodecahedron	23 $m\bar{3}$
432 $\bar{4}3m$	O T_d	24 24	Pentagon-trioctahedron Hexatetrahedron	Snub cube Cube truncated by two tetrahedra	432 $\bar{4}3m$
$m\bar{3}m, \frac{4}{m}\frac{2}{3}\frac{2}{m}$	O_h	48	Hexaoctahedron	Cube truncated by octahedron and by rhomb-dodecahedron	$m\bar{3}m$
Icosahedral system* (for details see Table 10.1.4.3)					
235 $m\bar{3}5, \frac{2}{m}\bar{3}5$	I I_h	60 120	Pentagon-hexecontahedron Hecatonicosahedron	Snub pentagon-dodecahedron Pentagon-dodecahedron truncated by icosahedron and by rhomb-triacontahedron	– –
Cylindrical system†					
∞ $\infty/m \equiv \bar{\infty}$ $\infty 2$	C_∞ $C_{\infty h} \equiv S_\infty \equiv C_{\infty i}$ D_∞	∞ ∞ ∞	Rotating cone Rotating double cone 'Anti-rotating' double cone	Rotating circle Rotating finite cylinder 'Anti-rotating' finite cylinder	– – –
∞m $\infty/mm \equiv \bar{\infty}m, \frac{\infty 2}{m m} \equiv \bar{\infty}\frac{2}{m}$	$C_{\infty v}$ $D_{\infty h} \equiv D_{\infty d}$	∞ ∞	Stationary cone Stationary double cone	Stationary circle Stationary finite cylinder	– –
Spherical system‡					
2∞ $m\bar{\infty}, \frac{2}{m}\bar{\infty}$	K K_h	∞ ∞	Rotating sphere Stationary sphere	Rotating sphere Stationary sphere	– –

* The Hermann–Mauguin symbols of the two icosahedral point groups are often written as 532 and $\bar{5}3m$ (see text).

† Rotating and 'anti-rotating' forms in the cylindrical system have no 'vertical' mirror planes, whereas stationary forms have infinitely many vertical mirror planes. In classes ∞ and $\infty 2$, enantiomorphism occurs, *i.e.* forms with opposite senses of rotation. Class $\infty/m \equiv \bar{\infty}$ exhibits no enantiomorphism due to the centre of symmetry, even though the double cone is rotating in one direction. This can be understood as follows: One single rotating cone can be regarded as a right-handed or left-handed screw, depending on the sense of rotation with respect to the axial direction from the base to the tip of the cone. Thus, the rotating double cone consists of two cones with opposite handedness and opposite orientations related by the (single) horizontal mirror plane. In contrast, the 'anti-rotating' double cone in class $\infty 2$ consists of two cones of equal handedness and opposite orientations, which are related by the (infinitely many) twofold axes. The term 'anti-rotating' means that upper and lower halves of the forms rotate in opposite directions.

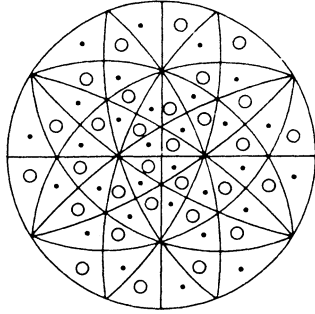
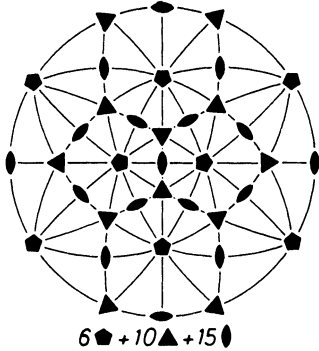
‡ The spheres in class 2∞ of the spherical system must rotate around an axis with at least two different orientations, in order to suppress all mirror planes. This class exhibits enantiomorphism, *i.e.* it contains spheres with either right-handed or left-handed senses of rotation around the axes (*cf.* Section 10.2.4, *Optical properties*). The stationary spheres in class $m\bar{\infty}$ contain infinitely many mirror planes through the centres of the spheres.

Group 2∞ is sometimes symbolized by $\infty\infty$; group $m\bar{\infty}$ by $\bar{\infty}\bar{\infty}$ or $\infty\infty m$. The symbols used here indicate the minimal symmetry necessary to generate the groups; they show, furthermore, the relation to the cubic groups. The Schoenflies symbol K is derived from the German name *Kugelgruppe*.

10. POINT GROUPS AND CRYSTAL CLASSES

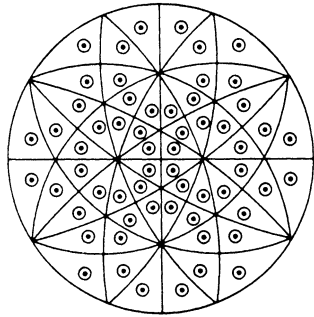
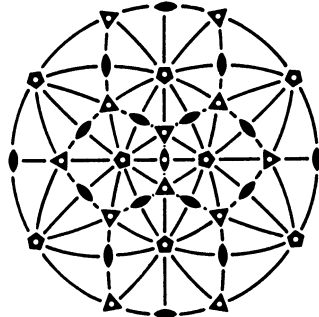
Table 10.1.4.3. *The two icosahedral point groups*

General, special and limiting face forms and point forms, oriented face- and site-symmetry symbols, and 'initial' values of (hkl) and x, y, z (see text).

235	I				
60	d	1	<p>Pentagon-hexecontahedron <i>Snub pentagon-dodecahedron</i> (= pentagon-dodecahedron + icosahedron + pentagon-hexecontahedron)</p> <p>Trisicosahedron <i>Pentagon-dodecahedron truncated by icosahedron</i> (poles between axes 2 and 3)</p> <p>Deltoid-hexecontahedron <i>Rhomb-triacontahedron & pentagon-dodecahedron & icosahedron</i> (poles between axes 3 and 5)</p> <p>Pentakisdodecahedron <i>Icosahedron truncated by pentagon-dodecahedron</i> (poles between axes 5 and 2)</p>	<p>(hkl) x, y, z</p> <p>$(0kl)$ with $l < 0.382 k$ $0, y, z$ with $z < 0.382 y$</p> <p>$(0kl)$ with $0.382 k < l < 1.618 k$ $0, y, z$ with $0.382 y < z < 1.618 y$</p> <p>$(0kl)$ with $l > 1.618 k$ $0, y, z$ with $z > 1.618 y$</p>	
30	c	2..	<p>Rhomb-triacontahedron <i>Icosadodecahedron</i> (= pentagon-dodecahedron & icosahedron)</p>	<p>(100) $x, 0, 0$</p>	
20	b	.3.	<p>Regular icosahedron <i>Regular pentagon-dodecahedron</i></p>	<p>(111) x, x, x</p>	
12	a	..5	<p>Regular pentagon-dodecahedron <i>Regular icosahedron</i></p>	<p>(01τ) $0, y, \tau y$ } with $\tau = \frac{1}{2}(\sqrt{5} + 1) = 1.618$</p>	
Symmetry of special projections					
		Along $[001]$	Along $[111]$	Along $[1\tau 0]$	
		$2mm$	$3m$	$5m$	

10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

Table 10.1.4.3. *The two icosahedral point groups (cont.)*

$m\bar{3}\bar{5}$	I_h			
				$6\blacklozenge + 10\blacktriangle + 15\bullet + 15m + \text{Centre}$
120	e	1	Hecatonicosahedron or hexaicosahedron <i>Pentagon-dodecahedron truncated by icosahedron and by rhomb-triacontahedron</i>	(hkl) x, y, z
60	d	$m..$	Trisicosahedron <i>Pentagon-dodecahedron truncated by icosahedron</i> (poles between axes 2 and $\bar{3}$)	$(0kl)$ with $ l < 0.382 k $ $0, y, z$ with $ z < 0.382 y $
			Deltoid-hexecontahedron <i>Rhomb-triacontahedron & pentagon-dodecahedron & icosahedron</i> (poles between axes $\bar{3}$ and $\bar{5}$)	$(0kl)$ with $0.382 k < l < 1.618 k $ $0, y, z$ with $0.382 y < z < 1.618 y $
			Pentakis-dodecahedron <i>Icosahedron truncated by pentagon-dodecahedron</i> (poles between axes $\bar{5}$ and 2)	$(0kl)$ with $ l > 1.618 k $ $0, y, z$ with $ z > 1.618 y $
30	c	$2mm..$	Rhomb-triacontahedron <i>Icosadodecahedron (= pentagon-dodecahedron & icosahedron)</i>	(100) $x, 0, 0$
20	b	$3m (m\bar{3}.)$	Regular icosahedron <i>Regular pentagon-dodecahedron</i>	(111) x, x, x
12	a	$5m (m.5)$	Regular pentagon-dodecahedron <i>Regular icosahedron</i>	(01τ) $0, y, \tau y$ } with $\tau = \frac{1}{2}(\sqrt{5} + 1) = 1.618$
Symmetry of special projections				
		Along $[001]$	Along $[111]$	Along $[1\tau 0]$
		$2mm$	$6mm$	$10mm$

10. POINT GROUPS AND CRYSTAL CLASSES

(b) the matrices Y^{-1}, Y^{-2}, Y^{-3} and Y^{-4} for the point coordinates.

This sequence of matrices ensures the same correspondence between the Miller indices and the point coordinates as for the crystallographic point groups in Table 10.1.2.2.

The matrices* are

$$Y = Y^{-4} = \begin{pmatrix} \frac{1}{2} & g & G \\ g & G & -\frac{1}{2} \\ -G & \frac{1}{2} & g \end{pmatrix}, \quad Y^2 = Y^{-3} = \begin{pmatrix} -g & G & \frac{1}{2} \\ G & \frac{1}{2} & -g \\ -\frac{1}{2} & g & -G \end{pmatrix},$$

$$Y^3 = Y^{-2} = \begin{pmatrix} -g & G & -\frac{1}{2} \\ G & \frac{1}{2} & g \\ \frac{1}{2} & -g & -G \end{pmatrix}, \quad Y^4 = Y^{-1} = \begin{pmatrix} \frac{1}{2} & g & -G \\ g & G & \frac{1}{2} \\ G & -\frac{1}{2} & g \end{pmatrix},$$

with†

$$G = \frac{\sqrt{5} + 1}{4} = \frac{\tau}{2} = \cos 36^\circ = 0.80902 \simeq \frac{72}{89}$$

$$g = \frac{\sqrt{5} - 1}{4} = \frac{\tau - 1}{2} = \cos 72^\circ = 0.30902 \simeq \frac{17}{55}.$$

These matrices correspond to counter-clockwise rotations of 72, 144, 216 and 288° around a fivefold axis parallel to $[1\tau 0]$.

The resulting indices h, k, l and coordinates x, y, z are irrational but can be approximated closely by rational (or integral) numbers. This explains the occurrence of almost regular icosahedra or pentagon-dodecahedra as crystal forms (for instance pyrite) or atomic groups (for instance B_{12} icosahedron).

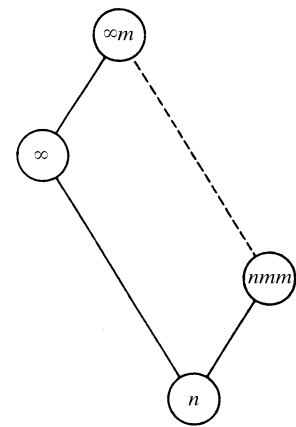
Further descriptions (including diagrams) of noncrystallographic groups are contained in papers by Nowacki (1933) and A. Niggli (1963) and in the textbooks by P. Niggli (1941, pp. 78–80, 96), Shubnikov & Koptsik (1974) and Vainshtein (1994). For the geometry of polyhedra, the well known books by H. S. M. Coxeter (especially Coxeter, 1973) are recommended.

10.1.4.3. Sub- and supergroups of the general point groups

In Figs. 10.1.4.1 to 10.1.4.3, the subgroup and supergroup relations between the two-dimensional and three-dimensional general point groups are illustrated. It should be remembered that the index of a group–subgroup relation between two groups of order infinity may be finite or infinite. For the two spherical groups, for instance, the index is $[2]$; the cylindrical groups, on the other hand, are subgroups of index $[\infty]$ of the spherical groups.

Fig. 10.1.4.1 for two dimensions shows that the two circular groups ∞m and ∞ have subgroups of types nmm and n , respectively, each of index $[\infty]$. The order of the rotation point may be $n = 4N, n = 4N + 2$ or $n = 2N + 1$. In the first case, the subgroups belong to the $4N$ -gonal system, in the latter two cases, they belong to the $(4N + 2)$ -gonal system. [In the diagram of the $(4N + 2)$ -gonal system, the $(2N + 1)$ -gonal groups appear with the symbols $\frac{1}{2}nm$ and $\frac{1}{2}n$.] The subgroups of the circular groups are not maximal because for any given value of N there exists a group with $N' = 2N$ which is both a subgroup of the circular group and a supergroup of the initial group.

The subgroup relations, for a specified value of N , within the $4N$ -gonal and the $(4N + 2)$ -gonal system, are shown in the lower part of the figure. They correspond to those of the crystallographic groups. A finite number of further maximal subgroups is obtained for lower



If $n = 4N$, go to $4N$ -gonal system
If $n = 4N + 2$ or $2N + 1$, go to $(4N + 2)$ -gonal system

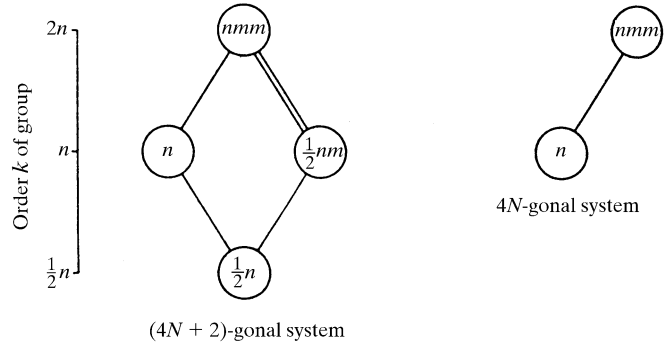


Fig. 10.1.4.1. Subgroups and supergroups of the two-dimensional general point groups. Solid lines indicate maximal normal subgroups, double solid lines mean that there are two maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. For the finite groups, the orders are given on the left. Note that the subgroups of the two circular groups are not maximal and the diagram applies only to a specified value of N (see text). For complete examples see Fig. 10.1.4.2.

values of N , until the crystallographic groups (Fig. 10.1.3.1) are reached. This is illustrated for the case $N = 4$ in Fig. 10.1.4.2.

Fig. 10.1.4.3 for three dimensions illustrates that the two spherical groups $2/m\infty$ and 2∞ each have one infinite set of cylindrical maximal conjugate subgroups, as well as one infinite set

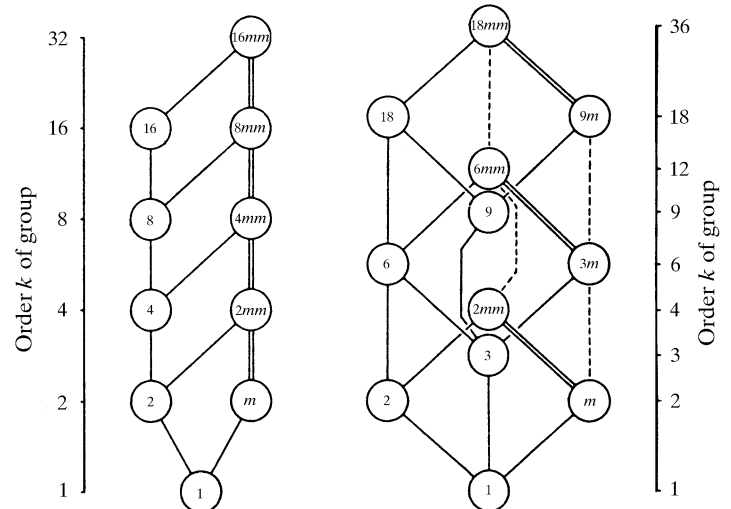


Fig. 10.1.4.2. The subgroups of the two-dimensional general point groups $16mm$ ($4N$ -gonal system) and $18mm$ [$(4N + 2)$ -gonal system including the $(2N + 1)$ -gonal groups]. Compare with Fig. 10.1.4.1 which applies only to one value of N .

* Note that for orthogonal matrices $Y^{-1} = Y^t$ (t = transposed).
† The number $\tau = 2G = 2g + 1 = 1.618034$ (Fibonacci number) is the characteristic value of the golden section $(\tau + 1) : \tau = \tau : 1$, i.e. $\tau(\tau - 1) = 1$. Furthermore, τ is the distance between alternating vertices of a regular pentagon of unit edge length and the distance from centre to vertex of a regular decagon of unit edge length.

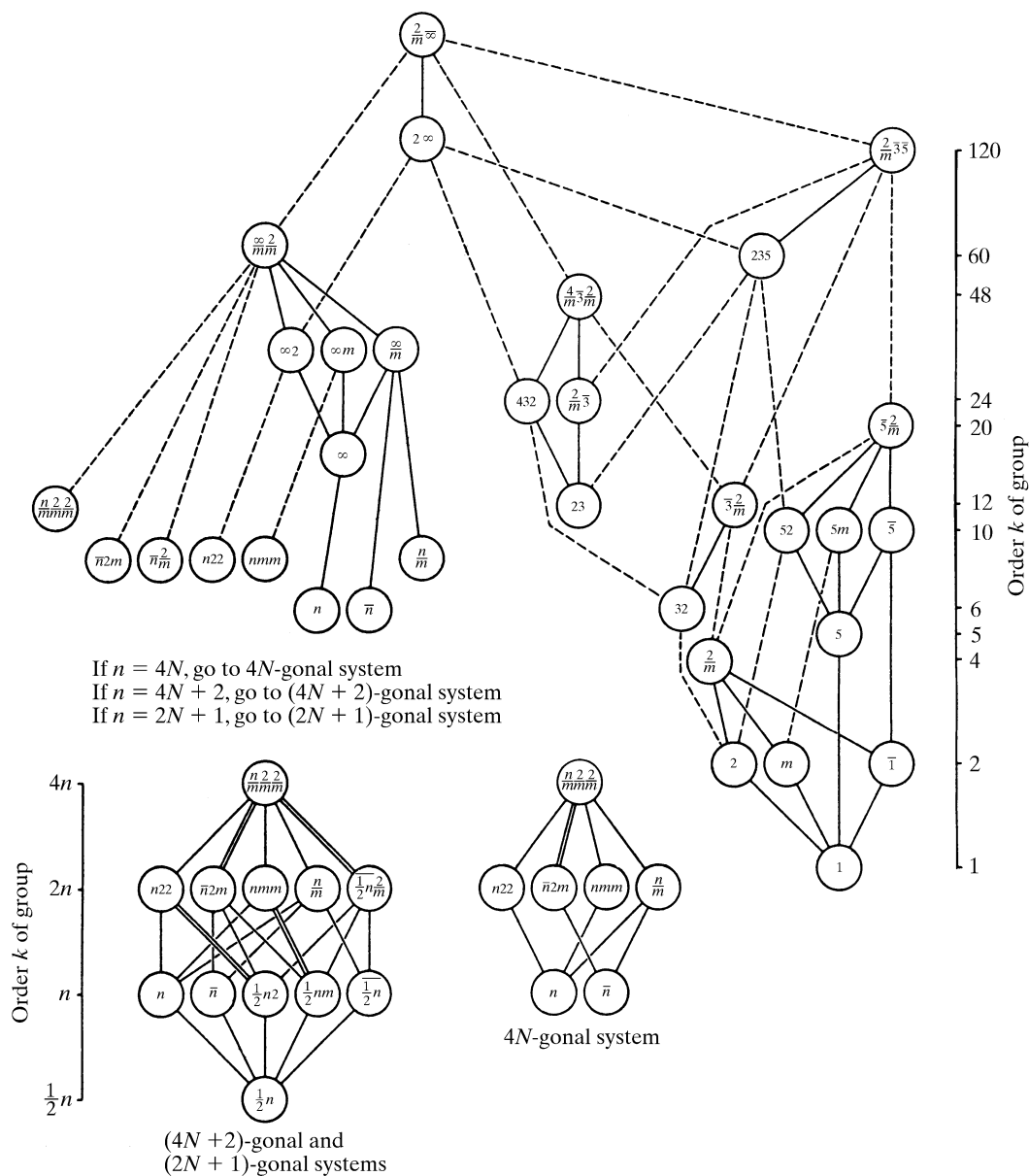


Fig. 10.1.4.3. Subgroups and supergroups of the three-dimensional general point groups. Solid lines indicate maximal normal subgroups, double solid lines mean that there are two maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. For the finite groups, the orders are given on the left and on the right. Note that the subgroups of the five cylindrical groups are not maximal and that the diagram applies only to a specified value of N (see text). Only those crystallographic point groups are included that are maximal subgroups of noncrystallographic point groups. Full Hermann–Mauguin symbols are used.

of cubic and one infinite set of icosahedral maximal finite conjugate subgroups, all of index $[\infty]$.

Each of the two icosahedral groups 235 and $2/m\bar{3}5$ has one set of five cubic, one set of six pentagonal and one set of ten trigonal maximal conjugate subgroups of indices $[5]$, $[6]$ and $[10]$, respectively (cf. Section 10.1.4.2, *The two icosahedral groups*); they are listed on the right of Fig. 10.1.4.3. For crystallographic groups, Fig. 10.1.3.2 applies. The subgroup types of the five cylindrical point groups are shown on the left of Fig. 10.1.4.3. As explained above for two dimensions, these subgroups are *not maximal* and of index $[\infty]$. Depending upon whether the main symmetry axis has the multiplicity $4N$, $4N + 2$ or $2N + 1$, the subgroups belong to the $4N$ -gonal, $(4N + 2)$ -gonal or $(2N + 1)$ -gonal system.

The subgroup and supergroup relations within these three systems are displayed in the lower left part of Fig. 10.1.4.3. They are analogous to the crystallographic groups. To facilitate the use of

the diagrams, the $(4N + 2)$ -gonal and the $(2N + 1)$ -gonal systems are combined, with the consequence that the five classes of the $(2N + 1)$ -gonal system now appear with the symbols $\frac{1}{2}n\frac{2}{m}$, $\frac{1}{2}n2$, $\frac{1}{2}nm$, $\frac{1}{2}n$ and $\frac{1}{2}n$. Again, the diagrams apply to a specified value of N . A finite number of further maximal subgroups is obtained for lower values of N , until the crystallographic groups (Fig. 10.1.3.2) are reached (cf. the two-dimensional examples in Fig. 10.1.4.2).

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