International Tables for Crystallography (2006). Vol. A, Section 13.1.2.1, pp. 836–838.

13.1. Isomorphic subgroups

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13.1.1. Definitions

A subgroup $\mathcal{H}$ of a space group $\mathcal{G}$ is an *isomorphic subgroup* if $\mathcal{H}$ is of the same or the enantiomorphic space-group type as $\mathcal{G}$. Thus, isomorphic space groups are a special subset of *klassengleiche subgroups*. The *maximal isomorphic subgroups of lowest index* are listed under **IIc** in the space-group tables of this volume (Part 7) (*cf.* Section 2.2.15). Isomorphic subgroups can easily be recognized because the standard space-group symbols of $\mathcal{G}$ and $\mathcal{H}$ are the same [isosymbolic subgroups (Billiet, 1973)] or the symbol of $\mathcal{H}$ is enantiomorphic to that of $\mathcal{G}$. Every space group has an infinite number of maximal isomorphic subgroups, whereas the number of maximal non-isomorphic subgroups is finite (*cf.* Section 8.3.3). For this reason, isomorphic subgroups are discussed in more detail in the present section.

If **a**, **b**, **c** are the basis vectors defining the conventional unit cell of $\mathcal{G}$ and **a**′, **b**′, **c**′ the basis vectors corresponding to $\mathcal{H}$ the relation

$$(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (\mathbf{a}, \mathbf{b}, \mathbf{c})\boldsymbol{S} \qquad (13.1.1.1)$$

holds, where $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ are row matrices and $\boldsymbol{S}$ is a (3×3) matrix. The coefficients S_{ij} of $\boldsymbol{S}$ are integers.*

The index of $\mathcal{H}$ in $\mathcal{G}$ is equal to $|\det(\boldsymbol{S})|$*, which is the ratio of the volumes $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$ and $[\mathbf{abc}]$ of the two cells. $\det(\boldsymbol{S})$ is positive if the bases of the two cells have the same handedness and negative if they have opposite handedness.

If O and O' are the origins of the coordinate systems $(O, \mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(O', \mathbf{a}', \mathbf{b}', \mathbf{c}')$, used for the description of $\mathcal{G}$ and $\mathcal{H}$, the column matrix of the coordinates of O' referred to the system $(O, \mathbf{a}, \mathbf{b}, \mathbf{c})$ will be denoted by $\boldsymbol{s}$. Thus, the coordinate system $(O', \mathbf{a}', \mathbf{b}', \mathbf{c}')$ will be specified completely by the square matrix $\boldsymbol{S}$ and the column matrix $\boldsymbol{s}$, symbolized by $\mathbb{S} : (\boldsymbol{S}, \boldsymbol{s})$.

An example of the application of equation (13.1.1.1) is given at the end of this chapter.

13.1.1.1. *The mathematical expression of equivalence*

Let $\mathbb{W} = (\boldsymbol{W}, \boldsymbol{w})$ be the operator of a given symmetry operation of $\mathcal{H}$ referred to $(O, \mathbf{a}, \mathbf{b}, \mathbf{c})$ and $\mathbb{W}' = (\boldsymbol{W}', \boldsymbol{w}')$ the operator of the same operation referred to $(O', \mathbf{a}', \mathbf{b}', \mathbf{c})$. Then the following relation applies

$$\mathbb{S}\mathbb{W}' = \mathbb{W}\mathbb{S} \quad \text{or} \quad \mathbb{W}' = \mathbb{S}^{-1}\mathbb{W}\mathbb{S} \qquad (13.1.1.2)$$

(*cf.* Bertaut & Billiet, 1979). The latter expression is more conventional, the former is easier to manipulate. Identifying the rotational (matrix) and translational (column) parts of $\mathbb{W}$, one obtains the following two conditions:

$$\begin{aligned} \boldsymbol{S}\boldsymbol{W}' &= \boldsymbol{W}\boldsymbol{S}, \\ \boldsymbol{s} + \boldsymbol{S}\boldsymbol{w}' &= \boldsymbol{w} + \boldsymbol{W}\boldsymbol{s} = \hat{\boldsymbol{w}} + \boldsymbol{t}_{\mathcal{G}} + \boldsymbol{W}\boldsymbol{s} \end{aligned} \qquad (13.1.1.2a)$$

or

$$\boldsymbol{S}\boldsymbol{w}' - \hat{\boldsymbol{w}} + (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{s} = \boldsymbol{t}_{\mathcal{G}}. \qquad (13.1.1.2b)$$

Here we have split $\boldsymbol{w}$ into a fractional part $\hat{\boldsymbol{w}}$ (smaller than any lattice translation) and $\boldsymbol{t}_{\mathcal{G}}$ which describes a lattice translation in $\mathcal{G}$.

The general expression of the matrix $\boldsymbol{S}$ is

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}. \qquad (13.1.1.3)$$

This general form, without any restrictions on the coefficients, applies only to the triclinic space groups $P1$ and $P\bar{1}$; $P1$ has only isomorphic subgroups (*cf.* Billiet, 1979; Billiet & Rolley Le Coz, 1980). For other space groups, restrictions have to be imposed on the coefficients S_{ij}.

13.1.2. Isomorphic subgroups

For convenience, we consider first those crystal systems that possess a unique direction (the privileged axis being taken parallel to **c**). We also include here the monoclinic system (unique axis either c or b).

13.1.2.1. *Monoclinic, tetragonal, trigonal, hexagonal systems*

If $\boldsymbol{W}$ is the matrix corresponding to a rotation about the c axis, $\boldsymbol{W}' = \boldsymbol{W}$ holds if the positive direction is the same for c and c'.† In consequence, $\boldsymbol{W}$ *must commute with* $\boldsymbol{S}$ [*cf.* equation (13.1.1.2*a*)]. This condition imposes relations on the coefficients S_{ij} of the matrix so that $\boldsymbol{S}$ and $\det(\boldsymbol{S})$ take the following forms:

Monoclinic system

$$\boldsymbol{M}_c = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{M}_c) = S_{33}(S_{11}S_{22} - S_{12}S_{21});$$

or if b instead of c is used

$$\boldsymbol{M}_b = \begin{pmatrix} S_{11} & 0 & S_{13} \\ 0 & S_{22} & 0 \\ S_{31} & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{M}_b) = S_{22}(S_{11}S_{33} - S_{13}S_{31}).$$

Tetragonal system

$$\boldsymbol{T}_1 = \begin{pmatrix} S_{11} & -S_{21} & 0 \\ S_{21} & S_{11} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{T}_1) = S_{33}(S_{11}^2 + S_{21}^2).$$

Hexagonal and trigonal systems

$$\boldsymbol{H}_1 = \begin{pmatrix} S_{11} & -S_{21} & 0 \\ S_{21} & S_{11} - S_{21} & 0 \\ 0 & 0 & S_{33} \end{pmatrix},$$

$$\det(\boldsymbol{H}_1) = S_{33}(S_{11}^2 + S_{21}^2 - S_{11}S_{21}).$$

For *rhombohedral* space groups, the matrix $\boldsymbol{H}_1$ applies only when hexagonal axes are used. If rhombohedral axes are used, the matrix $\boldsymbol{S}$ has the form

$$\boldsymbol{R}_1 = \begin{pmatrix} S_0 & S_2 & S_1 \\ S_1 & S_0 & S_2 \\ S_2 & S_1 & S_0 \end{pmatrix},$$

$$\begin{aligned} \det(\boldsymbol{R}_1) &= S_0^3 + S_1^3 + S_2^3 - 3S_0S_1S_2 \\ &= (S_0 + S_1 + S_2) \\ &\quad \times (S_0^2 + S_1^2 + S_2^2 - S_0S_1 - S_1S_2 - S_2S_0). \end{aligned}$$

* In general, this does not hold for non-isomorphic subgroups.

† If the positive directions of c and c' are opposite, $\boldsymbol{W}' = \boldsymbol{W}^{-1}$, but this does not bring in any new features.

Table 13.1.2.1. *Isomorphic subgroups of the plane groups*

OBLIQUE SYSTEM

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

Conditions: $S_{11} > 0, S_{22} > 0, S_{11}S_{22} > 1, S_{21} = 0, -S_{11}/2 < S_{12} \leq S_{11}/2$

RECTANGULAR SYSTEM

$$\boldsymbol{O} = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$$

Conditions: $S_{11} > 0, S_{22} > 0, S_{11}S_{22} > 1$

	$\boldsymbol{O}^a$	$\boldsymbol{O}^b$	$\boldsymbol{O}^c$	$\boldsymbol{O}^d$	$\boldsymbol{O}^e$
S_{11}	n_1	n_1	$2n_1 + 1$	$2n_1$	$2n_1 + 1$
S_{22}	n_2	$2n_2 + 1$	$2n_2 + 1$	$2n_2$	n_2

SQUARE SYSTEM

$$\boldsymbol{T}_1 = \begin{pmatrix} S_{11} & -S_{21} \\ S_{21} & S_{11} \end{pmatrix}$$

Conditions: $S_{11} > 0, S_{21} \geq 0, S_{11}^2 + S_{21}^2 > 1$

$$\boldsymbol{T}_2 = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{11} \end{pmatrix}$$

Conditions: $\boldsymbol{T}_2^a : S_{11} > 1;\ \boldsymbol{T}_2^b : S_{11} = 2n_1 + 1 > 1$

$$\boldsymbol{T}_3 = \begin{pmatrix} S_{11} & -S_{11} \\ S_{11} & S_{11} \end{pmatrix}$$

Condition: $S_{11} > 0$

HEXAGONAL SYSTEM

$$\boldsymbol{H}_1 = \begin{pmatrix} S_{11} & -S_{21} \\ S_{21} & S_{11} - S_{21} \end{pmatrix}$$

Conditions: $S_{11} > 0, 0 \leq S_{21} < S_{11}, (S_{11}^2 + S_{21}^2 - S_{11}S_{21}) > 1$

$$\boldsymbol{H}_2 = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{11} \end{pmatrix}$$

Condition: $S_{11} > 1$

$$\boldsymbol{H}_3 = \begin{pmatrix} 2S_{11} & -S_{11} \\ S_{11} & S_{11} \end{pmatrix}$$

Condition: $S_{11} > 0$

Table of plane subgroups

No. 1 *p*1 : $\boldsymbol{S}$; No. 2 *p*2 : $\boldsymbol{S}$; No. 3 *pm* : $\boldsymbol{O}^a$; No. 4 *pg* : $\boldsymbol{O}^b$;
No. 5 *cm* : $\boldsymbol{O}^c, \boldsymbol{O}^d$; No. 6 *p*2*mm* : $\boldsymbol{O}^a$; No. 7 *p*2*mg* : $\boldsymbol{O}^e$; No. 8 *p*2*gg* : $\boldsymbol{O}^c$;
No. 9 *c*2*mm* : $\boldsymbol{O}^c, \boldsymbol{O}^d$; No. 10 *p*4 : $\boldsymbol{T}_1$; No. 11 *p*4*mm* : $\boldsymbol{T}_2^a, \boldsymbol{T}_3$;
No. 12 *p*4*gm* : $\boldsymbol{T}_2^b$; No. 13 *p*3 : $\boldsymbol{H}_1$; No. 14 *p*3*m*1 : $\boldsymbol{H}_2$; No. 15 *p*31*m* : $\boldsymbol{H}_2$;
No. 16 *p*6 : $\boldsymbol{H}_1$; No. 17 *p*6*mm* : $\boldsymbol{H}_2, \boldsymbol{H}_3$.

13.1.2.1.1. *Additional restrictions*

If mirror or glide planes parallel to and/or twofold rotation or screw axes perpendicular to the principal rotation axis exist, further conditions are imposed upon the coefficients S_{ij} and these are indicated below (*cf.* Bertaut & Billiet, 1979).

Monoclinic system

The matrices $\boldsymbol{M}_c$ and $\boldsymbol{M}_b$ apply without any further restrictions on the coefficients.

Tetragonal system

The matrix $\boldsymbol{T}_1$ is valid for all space groups belonging to the crystal classes 4, $\bar{4}$ and $4/m$.

For all other space groups, restrictions apply to the coefficients S_{21} according to the following two rules which are consequences of equation (13.1.1.2*a*):

(i) If the last two letters of the Hermann–Mauguin symbol are different, $S_{21} = 0$; the corresponding matrix is called $\boldsymbol{T}_2$.

*Example: P*4$_2$*/mmc*

$$\boldsymbol{T}_2 = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{11} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{T}_2) = S_{33}S_{11}^2.$$

(ii) If the last two letters are the same (except for the three cases mentioned below), two matrices have to be applied, the matrix $\boldsymbol{T}_2$ introduced above and the matrix $\boldsymbol{T}_1$ with $S_{21} = S_{11}$; the corresponding matrix is called $\boldsymbol{T}_3$.

$$\boldsymbol{T}_3 = \begin{pmatrix} S_{11} & -S_{11} & 0 \\ S_{11} & S_{11} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{T}_3) = 2S_{33}S_{11}^2.$$

The following space groups have matrices $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$: *P*422, *P*4*mm*, *P*4/*mmm*, *P*4$_1$22, *P*4$_3$22, *P*4$_2$22, *P*4*cc*, *P*4/*mcc*, *I*422, *I*4*mm* and *I*4/*mmm*. The three exceptions to the rule mentioned above are the space groups *P*4/*nmm*, *P*4/*ncc* and *I*4$_1$22, which allow only $\boldsymbol{T}_2$.

Hexagonal and trigonal systems

The matrix $\boldsymbol{H}_1$ is valid for all space groups belonging to the crystal classes 6, $\bar{6}$, $6/m$, 3 and $\bar{3}$.

For all other space groups for which the last two letters of the Hermann–Mauguin symbol are different, $S_{22} = S_{11}$, and the matrix is called $\boldsymbol{H}_2$. Examples are *P*6$_3$/*mcm*, *P*312 and *P*$\bar{6}$2*m*.

$$\boldsymbol{H}_2 = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{11} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{H}_2) = S_{33}S_{11}^2.$$

If the last two letters of the Hermann–Mauguin symbol are the same, two matrices have to be applied, the matrix $\boldsymbol{H}_2$ introduced above and the matrix $\boldsymbol{H}_1$ with $S_{11} = 2S'_{11}$ and $S_{21} = S'_{11}$; this matrix is called $\boldsymbol{H}_3$,

$$\boldsymbol{H}_3 = \begin{pmatrix} 2S'_{11} & -S'_{11} & 0 \\ S'_{11} & S'_{11} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \quad \det(\boldsymbol{H}_3) = 3S_{33}S'^2_{11}.$$

Examples are *P*622, *P*6/*mmm* and *P*6*cc*.

Rhombohedral space groups

For *R*3 and *R*$\bar{3}$, one has the matrix $\boldsymbol{H}_1$ for hexagonal axes and $\boldsymbol{R}_1$ for rhombohedral axes. For all other rhombohedral space groups, one has $\boldsymbol{H}_2$ (hexagonal axes) and the matrix $\boldsymbol{R}_1$ with $S_1 = S_2$ (rhombohedral axes). This last matrix is called $\boldsymbol{R}_2$. Example: *R*32.

Table 13.1.2.2. *Isomorphic subgroups of the space groups*

TRICLINIC SYSTEM

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

Conditions: $S_{11} > 0, S_{22} > 0, S_{33} > 0, S_{11}S_{22}S_{33} > 1,$
$S_{21} = S_{31} = S_{32} = 0,\ -S_{11}/2 < S_{12} \le S_{11}/2,$
$-S_{11}/2 < S_{13} \le S_{11}/2,\ -S_{22}/2 < S_{23} \le S_{22}/2$

MONOCLINIC SYSTEM

Unique axis *c*

$$\boldsymbol{M}_c = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}$$

Conditions: $S_{33} > 0, (S_{11}S_{22} - S_{12}S_{21})S_{33} > 1$

	S_{11}	S_{12}	S_{21}	S_{22}	S_{33}	Extra condition
$\boldsymbol{M}_c^a$	$n_1 > 0$	0	n_3	$n_4 > 0$	n_5	$-n_4/2 < n_3 \le n_4/2$
$\boldsymbol{M}_c^b$	$n_1 > 0$	0	n_3	$n_4 > 0$	$2n_5 + 1$	$-n_4/2 < n_3 \le n_4/2$
$\boldsymbol{M}_c^c$	$n_1 > 0$	$2n_2$	0	$2n_4 + 1 > 0$	$2n_5 + 1$	$-n_1/2 < n_2 \le n_1/2$
$\boldsymbol{M}_c^d$	$n_1 > 0$	$2n_2$	0	$2n_4 > 0$	$2n_5$	$-n_1/2 < n_2 \le n_1/2$
$\boldsymbol{M}_c^e$	n_1	$2n_2 > 0$	$n_3 < 0$	0	$2n_5$	$-n_2 < n_1 \le n_2$
$\boldsymbol{M}_c^f$	$2n_1 + 1 > 0$	0	$2n_3$	$n_4 > 0$	n_5	$-n_4/2 < n_3 \le n_4/2$
$\boldsymbol{M}_c^g$	$2n_1 + 1 > 0$	$2n_2$	0	$2n_4 + 1 > 0$	$2n_5 + 1$	$-(2n_1+1)/2 < n_2 \le (2n_1+1)/2$
$\boldsymbol{M}_c^h$	$2n_1 + 1 > 0$	$2n_2$	0	$2n_4 > 0$	$2n_5$	$-(2n_1+1)/2 < n_2 \le (2n_1+1)/2$
$\boldsymbol{M}_c^i$	$2n_1 + 1$	$2n_2 > 0$	$n_3 < 0$	0	$2n_5$	$-(n_2+1)/2 < n_1 \le (n_2-1)/2$
$\boldsymbol{M}_c^j$	$2n_1 + 1 > 0$	0	$2n_3$	$n_4 > 0$	$2n_5 + 1$	$-n_4/2 < n_3 \le n_4/2$

Unique axis *b*

$$\boldsymbol{M}_b = \begin{pmatrix} S_{11} & 0 & S_{13} \\ 0 & S_{22} & 0 \\ S_{31} & 0 & S_{33} \end{pmatrix}$$

Conditions: $S_{22} > 0, (S_{11}S_{33} - S_{13}S_{31})S_{22} > 1$

	S_{11}	S_{13}	S_{22}	S_{31}	S_{33}	Extra condition
$\boldsymbol{M}_b^a$	$n_1 > 0$	n_2	n_3	0	$n_5 > 0$	$-n_1/2 < n_2 \le n_1/2$
$\boldsymbol{M}_b^b$	$n_1 > 0$	n_2	$2n_3 + 1$	0	$n_5 > 0$	$-n_1/2 < n_2 \le n_1/2$
$\boldsymbol{M}_b^c$	$2n_1 + 1 > 0$	0	$2n_3 + 1$	$2n_4$	$n_5 > 0$	$-n_5/2 < n_4 \le n_5/2$
$\boldsymbol{M}_b^d$	$2n_1 > 0$	0	$2n_3$	$2n_4$	$n_5 > 0$	$-n_5/2 < n_4 \le n_5/2$
$\boldsymbol{M}_b^e$	0	$n_2 < 0$	$2n_3$	$2n_4 > 0$	n_5	$-n_4 < n_5 \le n_4$
$\boldsymbol{M}_b^f$	n_1	$2n_2$	n_3	0	$2n_5 + 1 > 0$	$-n_1/2 < n_2 \le n_1/2$
$\boldsymbol{M}_b^g$	$2n_1 + 1 > 0$	0	$2n_3 + 1$	$2n_4$	$2n_5 + 1 > 0$	$-(2n_5+1)/2 < n_4 \le (2n_5+1)/2$
$\boldsymbol{M}_b^h$	$2n_1 > 0$	0	$2n_3$	$2n_4$	$2n_5 + 1 > 0$	$-(2n_5+1)/2 < n_4 \le (2n_5+1)/2$
$\boldsymbol{M}_b^i$	0	$n_2 < 0$	$2n_3$	$2n_4 > 0$	$2n_5 + 1$	$-(n_4+1)/2 < n_5 \le (n_4-1)/2$
$\boldsymbol{M}_b^j$	$n_1 > 0$	$2n_2$	$2n_3 + 1$	0	$2n_5 + 1 > 0$	$-n_1/2 < n_2 \le n_1/2$

$$\boldsymbol{R}_2 = \begin{pmatrix} S_0 & S_1 & S_1 \\ S_1 & S_0 & S_1 \\ S_1 & S_1 & S_0 \end{pmatrix}, \quad \det(\boldsymbol{R}_2) = (S_0 + 2S_1)(S_0 - S_1)^2.$$

13.1.2.2. *Cubic and orthorhombic systems*

Cubic system

For cubic space groups, equation (13.1.1.2*a*) leads to the matrix $\boldsymbol{C}$:

$$\boldsymbol{C} = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{pmatrix}, \quad \det(\boldsymbol{C}) = S^3.$$

Orthorhombic system

There are six choices of matrices $\boldsymbol{O}_i$ $(i = 1, 2, 3, 4, 5, 6)$ corresponding to the identical orientation ($\boldsymbol{O}_1$), to cyclic permutations of the three axes ($\boldsymbol{O}_2$ and $\boldsymbol{O}_3$) and to the interchange of two axes ($\boldsymbol{O}_4, \boldsymbol{O}_5$ and $\boldsymbol{O}_6$), *i.e.* to the six orthorhombic 'settings'.

$$\boldsymbol{O}_1 = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}; \quad \boldsymbol{O}_2 = \begin{pmatrix} 0 & S_{12} & 0 \\ 0 & 0 & S_{23} \\ S_{31} & 0 & 0 \end{pmatrix};$$

$$\boldsymbol{O}_3 = \begin{pmatrix} 0 & 0 & S_{13} \\ S_{21} & 0 & 0 \\ 0 & S_{32} & 0 \end{pmatrix}; \quad \boldsymbol{O}_4 = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{32} & 0 \end{pmatrix};$$

$$\boldsymbol{O}_5 = \begin{pmatrix} 0 & 0 & S_{13} \\ 0 & S_{22} & 0 \\ S_{31} & 0 & 0 \end{pmatrix}; \quad \boldsymbol{O}_6 = \begin{pmatrix} 0 & S_{12} & 0 \\ S_{21} & 0 & 0 \\ 0 & 0 & S_{33} \end{pmatrix}.$$

The determinant is always equal to the product of the three nonzero coefficients, $\det(\boldsymbol{O}_i) = \pm S_{1j}S_{2k}S_{3l}$.

references