International Tables for Crystallography (2006). Vol. B, Figure 4.6.3.9, p. 502.

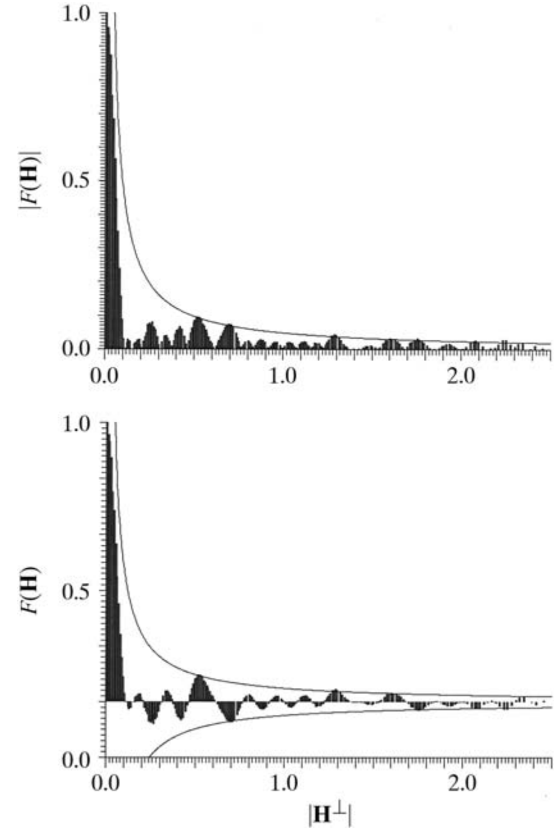

Fig. 4.6.3.9. The structure factors $F(\mathbf{H})$ (below) and their magnitudes $|F(\mathbf{H})|$ (above) of the squared Fibonacci chain decorated with equal point atoms are shown as a function of the perpendicular-space component $|\mathbf{H}^{\perp}|$ of the diffraction vector. The short distance is $\mathrm{S} = 2.5$ Å, all structure factors within $0 \leq |\mathbf{H}| \leq 2.5$ Å^{-1} have been calculated and normalized to $F(00) = 1$.

Further scaling relationships in reciprocal space exist: scaling a diffraction vector $\mathbf{H} = h_1\mathbf{d}_1^* + h_2\mathbf{d}_2^* = h_1 a^* \begin{pmatrix} 1 \\ -\tau \end{pmatrix}_V + h_2 a^* \begin{pmatrix} \tau \\ 1 \end{pmatrix}_V$ with the matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_D$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_D \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_D = \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix}_D \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_D = \begin{pmatrix} F_n h_1 + F_{n+1} h_2 \\ F_{n+1} h_1 + F_{n+2} h_2 \end{pmatrix}_D,$$

increases the magnitudes of structure factors assigned to this particular diffraction vector $\mathbf{H}$,

$$|F(S^n\mathbf{H})| > |F(S^{n-1}\mathbf{H})| > \ldots > |F(S\mathbf{H})| > |F(\mathbf{H})|.$$

This is due to the shrinking of the perpendicular-space component of the diffraction vector by powers of $(-\tau)^{-n}$ while expanding the parallel-space component by τ^n according to the eigenvalues τ and $-\tau^{-1}$ of S acting in the two eigenspaces $\mathbf{V}^{\parallel}$ and $\mathbf{V}^{\perp}$:

$$\begin{aligned}\pi^{\parallel}(S\mathbf{H}) &= (h_2 + \tau(h_1 + h_2))a^* = (\tau h_1 + h_2(\tau + 1))a^* \\ &= \tau(h_1 + \tau h_2)a^*, \\ \pi^{\perp}(S\mathbf{H}) &= (-\tau h_2 + h_1 + h_2)a^* = (h_1 - h_2(\tau - 1))a^* \\ &= -(1/\tau)(-\tau h_1 + h_2)a^*,\end{aligned}$$

$$|F(\tau^n\mathbf{H}^{\parallel})| > |F(\tau^{n-1}\mathbf{H}^{\parallel})| > \ldots > |F(\tau\mathbf{H}^{\parallel})| > |F(\mathbf{H}^{\parallel})|.$$

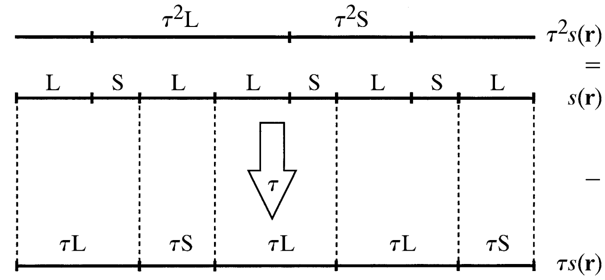

Fig. 4.6.3.10. Part . . . LSLLSLSL . . . of a Fibonacci sequence $s(\mathbf{r})$ before and after scaling by the factor τ. L is mapped onto τL, S onto $\tau\mathrm{S} = \mathrm{L}$. The vertices of the new sequence are a subset of those of the original sequence (the correspondence is indicated by dashed lines). The residual vertices $\tau^2 s(\mathbf{r})$, which give when decorating $\tau s(\mathbf{r})$ the Fibonacci sequence $s(\mathbf{r})$, form a Fibonacci sequence scaled by a factor τ^2.

Thus, for scaling n times we obtain

$$\begin{aligned}\pi^{\perp}(S^n\mathbf{H}) &= (-\tau(F_n h_1 + F_{n+1} h_2) + (F_{n+1} h_1 + F_{n+2} h_2))a^* \\ &= (h_1(-\tau F_n + F_{n+1}) + h_2(-\tau F_{n+1} + F_{n+2}))a^*\end{aligned}$$

with

$$\lim_{n\to\infty}(-\tau F_n + F_{n+1}) = 0 \text{ and } \lim_{n\to\infty}(-\tau F_{n+1} + F_{n+2}) = 0,$$

yielding eventually

$$\lim_{n\to\infty}(\pi^{\perp}(S^n\mathbf{H})) = 0 \text{ and } \lim_{n\to\infty}(F(S^n\mathbf{H})) = F(\mathbf{0}).$$

The scaling of the diffraction vectors $\mathbf{H}$ by S^n corresponds to a hyperbolic rotation (Janner, 1992) with angle $n\varphi$, where $\sinh\varphi = 1/2$ (Fig. 4.6.3.11):

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} = \begin{pmatrix} \cosh 2n\varphi & \sinh 2n\varphi \\ \sinh 2n\varphi & \cosh 2n\varphi \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n+1} = \begin{pmatrix} \sinh[(2n+1)\varphi] & \cosh[(2n+1)\varphi] \\ \cosh[(2n+1)\varphi] & \sinh[(2n+1)\varphi] \end{pmatrix}.$$

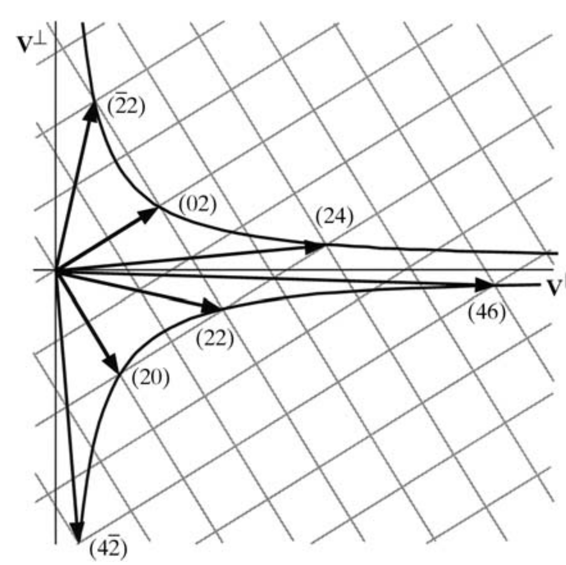

Fig. 4.6.3.11. Scaling operations of the Fibonacci sequence. The scaling operation S acts six times on the diffraction vector $\mathbf{H} = (4\bar{2})$ yielding the sequence $(4\bar{2}) \to (\bar{2}2) \to (20) \to (02) \to (22) \to (24) \to (46)$.

references