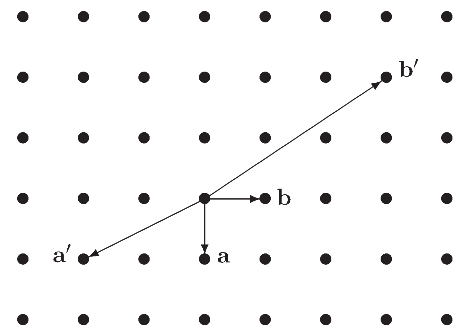

Figure 1.3.2.1
Conventional basis **a**, **b** and a non-conventional basis **a**′, **b**′ for the square lattice.

Example

The square lattice

$$\mathbf{L} = \mathbb{Z}^2 = \left\{ \begin{pmatrix} m \\ n \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}$$

in $\mathbb{V}^2$ has the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as its standard lattice basis. But

$$\mathbf{a}' = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{b}' = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

is also a lattice basis of **L**: on the one hand **a**′ and **b**′ are integral linear combinations of **a**, **b** and are thus contained in **L**. On the other hand

$$-3\mathbf{a}' - 2\mathbf{b}' = \begin{pmatrix} -3 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{a}$$

and

$$-2\mathbf{a}' - \mathbf{b}' = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{b},$$

hence **a** and **b** are also integral linear combinations of **a**′, **b**′ and thus the two bases **a**, **b** and **a**′, **b**′ both span the same lattice (see Fig. 1.3.2.1).

The example indicates how the different lattice bases of a lattice **L** can be described. Recall that for a vector $\mathbf{v} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ the coefficients x, y, z are called the *coordinates* and the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is called the *coordinate column* of **v** with respect to the basis **a**, **b**, **c**. The coordinate columns of the vectors in **L** with respect to a lattice basis are therefore simply columns with three integral components. In particular, if we take a second lattice basis **a**′, **b**′, **c**′ of **L**, then the coordinate columns of **a**′, **b**′, **c**′with respect to the first basis are columns of integers and thus the basis transformation $\boldsymbol{P}$ such that $(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (\mathbf{a}, \mathbf{b}, \mathbf{c})\boldsymbol{P}$ is an integral 3×3 matrix. But if we interchange the roles of the two bases, they are related by the inverse transformation $\boldsymbol{P}^{-1}$, *i.e.* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}', \mathbf{b}', \mathbf{c}')\boldsymbol{P}^{-1}$, and the argument given above asserts that $\boldsymbol{P}^{-1}$ is also an integral matrix. Now, on the one hand $\det \boldsymbol{P}$ and $\det \boldsymbol{P}^{-1}$ are both integers (being determinants of integral matrices), on the other hand $\det \boldsymbol{P}^{-1} = 1/\det \boldsymbol{P}$. This is only possible if $\det \boldsymbol{P} = \pm 1$.

Summarizing, the different lattice bases of a lattice **L** are obtained by transforming a single lattice basis **a**, **b**, **c** with integral transformation matrices $\boldsymbol{P}$ such that $\det \boldsymbol{P} = \pm 1$.

1.3.2.2. Metric properties

In the three-dimensional vector space $\mathbb{V}^3$, the *norm* or *length* of a vector $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ is (due to Pythagoras' theorem) given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

From this, the *scalar product*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z \text{ for } \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

is derived, which allows one to express angles by

$$\cos \angle(\mathbf{v}, \mathbf{w}) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

The definition of a norm function for the vectors turns $\mathbb{V}^3$ into a *Euclidean space*. A lattice **L** that is contained in $\mathbb{V}^3$ inherits the metric properties of this space. But for the lattice, these properties are most conveniently expressed with respect to a lattice basis. It is customary to choose basis vectors **a**, **b**, **c** which define a right-handed coordinate system, *i.e.* such that the matrix with columns **a**, **b**, **c** has a positive determinant.

Definition

For a lattice $\mathbf{L} \subseteq \mathbb{V}^3$ with lattice basis **a**, **b**, **c** the *metric tensor* of **L** is the 3×3 matrix

$$\boldsymbol{G} = \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{pmatrix}.$$

If $\boldsymbol{A}$ is the 3×3 matrix with the vectors **a**, **b**, **c** as its columns, then the metric tensor is obtained as the matrix product $\boldsymbol{G} = \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{A}$. It follows immediately that the metric tensor is a symmetric matrix, *i.e.* $\boldsymbol{G}^{\mathrm{T}} = \boldsymbol{G}$.

Example

Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

be the basis of a lattice **L**. Then the metric tensor of **L** (with respect to the given basis) is

$$\boldsymbol{G} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

With the help of the metric tensor the scalar products of arbitrary vectors, given as linear combinations of the lattice basis, can be computed from their coordinate columns as follows: If $\mathbf{v} = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}$ and $\mathbf{w} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$, then

references