*International Tables for Crystallography* (2006). Vol. B, Section 1.3.2.3.5, p. 30.



fundamental system $S$ of neighbourhoods of the origin in $\mathscr{E}(\Omega)$ is given by subsets of $\mathscr{E}(\Omega)$ of the form

$$V(m, \varepsilon, K) = \{\varphi \in \mathscr{E}(\Omega) | |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p},K}(\varphi) < \varepsilon\}$$

for all natural integers $m$, positive real $\varepsilon$, and compact subset $K$ of $\Omega$. Since a *countable* family of compact subsets $K$ suffices to cover $\Omega$, and since restricted values of $\varepsilon$ of the form $\varepsilon = 1/N$ lead to the same topology, $S$ is equivalent to a *countable* system of neighbourhoods and hence $\mathscr{E}(\Omega)$ *is metrizable*.

Convergence in $\mathscr{E}$ may thus be defined by means of sequences. A sequence $(\varphi_\nu)$ in $\mathscr{E}$ will be said to converge to 0 if for any given $V(m, \varepsilon, K)$ there exists $\nu_0$ such that $\varphi_\nu \in V(m, \varepsilon, K)$ whenever $\nu > \nu_0$; in other words, if the $\varphi_\nu$ and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly on any given compact $K$ in $\Omega$.

1.3.2.3.3.2. *Topology on* $\mathscr{D}_k(\Omega)$

It is defined by the family of semi-norms

$$\varphi \in \mathscr{D}_K(\Omega) \longmapsto \sigma_{\mathbf{p}}(\varphi) = \sup_{\mathbf{x} \in K} |D^{\mathbf{p}}\varphi(\mathbf{x})|,$$

where $K$ is now fixed. The fundamental system $S$ of neighbourhoods of the origin in $\mathscr{D}_K$ is given by sets of the form

$$V(m, \varepsilon) = \{\varphi \in \mathscr{D}_K(\Omega) | |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p}}(\varphi) < \varepsilon\}.$$

It is equivalent to the countable subsystem of the $V(m, 1/N)$, hence $\mathscr{D}_K(\Omega)$ *is metrizable*.

Convergence in $\mathscr{D}_K$ may thus be defined by means of sequences. A sequence $(\varphi_\nu)$ in $\mathscr{D}_K$ will be said to converge to 0 if for any given $V(m, \varepsilon)$ there exists $\nu_0$ such that $\varphi_\nu \in V(m, \varepsilon)$ whenever $\nu > \nu_0$; in other words, if the $\varphi_\nu$ and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly in $K$.

1.3.2.3.3.3. *Topology on* $\mathscr{D}(\Omega)$

It is defined by the fundamental system of neighbourhoods of the origin consisting of sets of the form

$$V((m), (\varepsilon)) = \left\{\varphi \in \mathscr{D}(\Omega) | |\mathbf{p}| \leq m_\nu \Rightarrow \sup_{\|\mathbf{x}\| \leq \nu} |D^{\mathbf{p}}\varphi(\mathbf{x})| < \varepsilon_\nu \text{ for all } \nu\right\},$$

where $(m)$ is an increasing sequence $(m_\nu)$ of integers tending to $+\infty$ and $(\varepsilon)$ is a decreasing sequence $(\varepsilon_\nu)$ of positive reals tending to 0, as $\nu \to \infty$.

This topology is *not metrizable*, because the sets of sequences $(m)$ and $(\varepsilon)$ are essentially uncountable. It can, however, be shown to be the *inductive limit* of the topology of the subspaces $\mathscr{D}_K$, in the following sense: $V$ is a neighbourhood of the origin in $\mathscr{D}$ if and only if its intersection with $\mathscr{D}_K$ is a neighbourhood of the origin in $\mathscr{D}_K$ for any given compact $K$ in $\Omega$.

A sequence $(\varphi_\nu)$ in $\mathscr{D}$ will thus be said to converge to 0 in $\mathscr{D}$ if all the $\varphi_\nu$ belong to some $\mathscr{D}_K$ (with $K$ a compact subset of $\Omega$ independent of $\nu$) and if $(\varphi_\nu)$ converges to 0 in $\mathscr{D}_K$.

As a result, a complex-valued functional $T$ on $\mathscr{D}$ will be said to be continuous for the topology of $\mathscr{D}$ if and only if, for any given compact $K$ in $\Omega$, its restriction to $\mathscr{D}_K$ is continuous for the topology of $\mathscr{D}_K$, *i.e.* maps convergent sequences in $\mathscr{D}_K$ to convergent sequences in $\mathbb{C}$.

This property of $\mathscr{D}$, *i.e.* having a non-metrizable topology which is the inductive limit of metrizable topologies in its subspaces $\mathscr{D}_K$, conditions the whole structure of distribution theory and dictates that of many of its proofs.

1.3.2.3.3.4. *Topologies on* $\mathscr{E}^{(m)}, \mathscr{D}_k^{(m)}, \mathscr{D}^{(m)}$

These are defined similarly, but only involve conditions on derivatives up to order $m$.

1.3.2.3.4. *Definition of distributions*

A *distribution* $T$ on $\Omega$ is a *linear form* over $\mathscr{D}(\Omega)$, *i.e.* a map

$$T : \varphi \longmapsto \langle T, \varphi\rangle$$

which associates linearly a complex number $\langle T, \varphi\rangle$ to any $\varphi \in \mathscr{D}(\Omega)$, and which is *continuous* for the topology of that space. In the terminology of Section 1.3.2.2.6.2, $T$ is an element of $\mathscr{D}'(\Omega)$, the *topological dual* of $\mathscr{D}(\Omega)$.

Continuity over $\mathscr{D}$ is equivalent to continuity over $\mathscr{D}_K$ for all compact $K$ contained in $\Omega$, and hence to the condition that for any sequence $(\varphi_\nu)$ in $\mathscr{D}$ such that

(i) Supp $\varphi_\nu$ is contained in some compact $K$ independent of $\nu$,

(ii) the sequences $(|D^{\mathbf{p}}\varphi_\nu|)$ converge uniformly to 0 on $K$ for all multi-indices $\mathbf{p}$;

*then* the sequence of complex numbers $\langle T, \varphi_\nu\rangle$ converges to 0 in $\mathbb{C}$.

If the continuity of a distribution $T$ requires (ii) for $|\mathbf{p}| \leq m$ only, $T$ may be defined over $\mathscr{D}^{(m)}$ and thus $T \in \mathscr{D}'^{(m)}$; $T$ is said to be a *distribution of finite order m*. In particular, for $m = 0$, $\mathscr{D}^{(0)}$ is the space of continuous functions with compact support, and a distribution $T \in \mathscr{D}'^{(0)}$ is a (Radon) *measure* as used in the theory of integration. Thus measures are particular cases of distributions.

Generally speaking, the *larger* a space of test functions, the *smaller* its topological dual:

$$m < n \Rightarrow \mathscr{D}^{(m)} \supset \mathscr{D}^{(n)} \Rightarrow \mathscr{D}'^{(n)} \supset \mathscr{D}'^{(m)}.$$

This clearly results from the observation that if the $\varphi$'s are allowed to be less regular, then less wildness can be accommodated in $T$ if the continuity of the map $\varphi \longmapsto \langle T, \varphi\rangle$ with respect to $\varphi$ is to be preserved.

1.3.2.3.5. *First examples of distributions*

(i) The linear map $\varphi \longmapsto \langle \delta, \varphi\rangle = \varphi(\mathbf{0})$ is a measure (*i.e.* a zeroth-order distribution) called Dirac's measure or (improperly) Dirac's '$\delta$-function'.

(ii) The linear map $\varphi \longmapsto \langle \delta_{(\mathbf{a})}, \varphi\rangle = \varphi(\mathbf{a})$ is called Dirac's measure at point $\mathbf{a} \in \mathbb{R}^n$.

(iii) The linear map $\varphi \longmapsto (-1)^{\mathbf{p}} D^{\mathbf{p}}\varphi(\mathbf{a})$ is a distribution of order $m = |\mathbf{p}| > 0$, and hence is not a measure.

(iv) The linear map $\varphi \longmapsto \sum_{\nu>0} \varphi^{(\nu)}(\nu)$ is a distribution of infinite order on $\mathbb{R}$: the order of differentiation is bounded for each $\varphi$ (because $\varphi$ has compact support) but is not as $\varphi$ varies.

(v) If $(\mathbf{p}_\nu)$ is a sequence of multi-indices $\mathbf{p}_\nu = (p_{1\nu}, \ldots, p_{n\nu})$ such that $|\mathbf{p}_\nu| \to \infty$ as $\nu \to \infty$, then the linear map $\varphi \longmapsto \sum_{\nu>0} (D^{\mathbf{p}_\nu}\varphi)(\mathbf{p}_\nu)$ is a distribution of infinite order on $\mathbb{R}^n$.

1.3.2.3.6. *Distributions associated to locally integrable functions*

Let $f$ be a complex-valued function over $\Omega$ such that $\int_K |f(\mathbf{x})| \,\mathrm{d}^n\mathbf{x}$ exists for any given compact $K$ in $\Omega$; $f$ is then called *locally integrable*.

The linear mapping from $\mathscr{D}(\Omega)$ to $\mathbb{C}$ defined by

$$\varphi \longmapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}$$

may then be shown to be continuous over $\mathscr{D}(\Omega)$. It thus defines a *distribution* $T_f \in \mathscr{D}'(\Omega)$:

$$\langle T_f, \varphi\rangle = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}.$$

As the continuity of $T_f$ only requires that $\varphi \in \mathscr{D}^{(0)}(\Omega)$, $T_f$ is actually a Radon measure.





references