$$\mathscr{F}[\delta_{\mathbf{x}}^{(\mathbf{m})}]_{\boldsymbol{\xi}} = (2\pi i\boldsymbol{\xi})^{\mathbf{m}}, \quad \mathscr{F}[\mathbf{x}^{\mathbf{m}}]_{\boldsymbol{\xi}} = (-2\pi i)^{-|\mathbf{m}|}\delta_{\boldsymbol{\xi}}^{(\mathbf{m})};$$
$$\mathscr{F}[\delta_{\mathbf{a}}]_{\boldsymbol{\xi}} = \exp(-2\pi i\mathbf{a}\cdot\boldsymbol{\xi}), \quad \mathscr{F}[\exp(2\pi i\boldsymbol{\alpha}\cdot\mathbf{x})]_{\boldsymbol{\xi}} = \delta_{\boldsymbol{\alpha}},$$

with analogous relations for $\bar{\mathscr{F}}$, i becoming $-i$. Thus derivatives of δ are mapped to monomials (and *vice versa*), while translates of δ are mapped to 'phase factors' (and *vice versa*).

1.3.2.5.7. *Reciprocity theorem*

The previous results now allow a self-contained and rigorous proof of the reciprocity theorem between $\mathscr{F}$ and $\bar{\mathscr{F}}$ to be given, whereas in traditional settings (*i.e.* in L^1 and L^2) the implicit handling of δ through a limiting process is always the sticking point.

Reciprocity is first established in $\mathscr{S}$ as follows:

$$\begin{aligned}\bar{\mathscr{F}}[\mathscr{F}[\varphi]](\mathbf{x}) &= \int_{\mathbb{R}^n} \mathscr{F}[\varphi](\boldsymbol{\xi})\exp(2\pi i\boldsymbol{\xi}\cdot\mathbf{x})\,\mathrm{d}^n\boldsymbol{\xi}\\ &= \int_{\mathbb{R}^n} \mathscr{F}[\tau_{-\mathbf{x}}\varphi](\boldsymbol{\xi})\,\mathrm{d}^n\boldsymbol{\xi}\\ &= \langle 1, \mathscr{F}[\tau_{-\mathbf{x}}\varphi]\rangle\\ &= \langle \mathscr{F}[1], \tau_{-\mathbf{x}}\varphi\rangle\\ &= \langle \tau_{\mathbf{x}}\delta, \varphi\rangle\\ &= \varphi(\mathbf{x})\end{aligned}$$

and similarly

$$\mathscr{F}[\bar{\mathscr{F}}[\varphi]](\mathbf{x}) = \varphi(\mathbf{x}).$$

The reciprocity theorem is then proved in $\mathscr{S}'$ by transposition:

$$\bar{\mathscr{F}}[\mathscr{F}[T]] = \mathscr{F}[\bar{\mathscr{F}}[T]] = T \quad \text{for all } T \in \mathscr{S}'.$$

Thus the Fourier cotransformation $\bar{\mathscr{F}}$ in $\mathscr{S}'$ may legitimately be called the 'inverse Fourier transformation'.

The method of Section 1.3.2.4.3 may then be used to show that $\mathscr{F}$ and $\bar{\mathscr{F}}$ both have period 4 in $\mathscr{S}'$.

1.3.2.5.8. *Multiplication and convolution*

Multiplier functions $\alpha(\mathbf{x})$ for tempered distributions must be infinitely differentiable, as for ordinary distributions; furthermore, they must grow sufficiently slowly as $\|x\| \to \infty$ to ensure that $\alpha\varphi \in \mathscr{S}$ for all $\varphi \in \mathscr{S}$ and that the map $\varphi \longmapsto \alpha\varphi$ is continuous for the topology of $\mathscr{S}$. This leads to choosing for multipliers the subspace $\mathscr{O}_M$ consisting of functions $\alpha \in \mathscr{E}$ *of polynomial growth*. It can be shown that if f is in $\mathscr{O}_M$, then the associated distribution T_f is in $\mathscr{S}'$ (*i.e.* is a tempered distribution); and that conversely if T is in $\mathscr{S}'$, $\mu * T$ is in $\mathscr{O}_M$ for all $\mu \in \mathscr{D}$.

Corresponding restrictions must be imposed to define the space $\mathscr{O}'_C$ of those distributions T whose convolution $S * T$ with a tempered distribution S is still a tempered distribution: T must be such that, for all $\varphi \in \mathscr{S}, \theta(\mathbf{x}) = \langle T_{\mathbf{y}}, \varphi(\mathbf{x}+\mathbf{y})\rangle$ is in $\mathscr{S}$; and such that the map $\varphi \longmapsto \theta$ be continuous for the topology of $\mathscr{S}$. This implies that S is 'rapidly decreasing'. It can be shown that if f is in $\mathscr{S}$, then the associated distribution T_f is in $\mathscr{O}'_C$; and that conversely if T is in $\mathscr{O}'_C, \mu * T$ is in $\mathscr{S}$ for all $\mu \in \mathscr{D}$.

The two spaces $\mathscr{O}_M$ and $\mathscr{O}'_C$ are mapped into each other by the Fourier transformation

$$\begin{aligned}\mathscr{F}(\mathscr{O}_M) &= \bar{\mathscr{F}}(\mathscr{O}_M) = \mathscr{O}'_C\\ \mathscr{F}(\mathscr{O}'_C) &= \bar{\mathscr{F}}(\mathscr{O}'_C) = \mathscr{O}_M\end{aligned}$$

and the convolution theorem takes the form

$$\begin{aligned}\mathscr{F}[\alpha S] &= \mathscr{F}[\alpha] * \mathscr{F}[S] \quad && S \in \mathscr{S}', \alpha \in \mathscr{O}_M, \mathscr{F}[\alpha] \in \mathscr{O}'_C;\\ \mathscr{F}[S * T] &= \mathscr{F}[S] \times \mathscr{F}[T] \quad && S \in \mathscr{S}', T \in \mathscr{O}'_C, \mathscr{F}[T] \in \mathscr{O}_M.\end{aligned}$$

The same identities hold for $\bar{\mathscr{F}}$. Taken together with the reciprocity theorem, these show that $\mathscr{F}$ and $\bar{\mathscr{F}}$ establish mutually inverse isomorphisms between $\mathscr{O}_M$ and $\mathscr{O}'_C$, and exchange multiplication for convolution in $\mathscr{S}'$.

It may be noticed that most of the basic properties of $\mathscr{F}$ and $\bar{\mathscr{F}}$ may be deduced from this theorem and from the properties of δ. Differentiation operators $D^{\mathbf{m}}$ and translation operators $\tau_{\mathbf{a}}$ are convolutions with $D^{\mathbf{m}}\delta$ and $\tau_{\mathbf{a}}\delta$; they are turned, respectively, into multiplication by monomials $(\pm 2\pi i\boldsymbol{\xi})^{\mathbf{m}}$ (the transforms of $D^{\mathbf{m}}\delta$) or by phase factors $\exp(\pm 2\pi i\boldsymbol{\xi}\cdot\boldsymbol{\alpha})$ (the transforms of $\tau_{\mathbf{a}}\delta$).

Another consequence of the convolution theorem is the duality established by the Fourier transformation between sections and projections of a function and its transform. For instance, in $\mathbb{R}^3$, the *projection* of $f(x, y, z)$ on the x, y plane along the z axis may be written

$$(\delta_x \otimes \delta_y \otimes 1_z) * f;$$

its Fourier transform is then

$$(1_\xi \otimes 1_\eta \otimes \delta_\zeta) \times \mathscr{F}[f],$$

which is the *section* of $\mathscr{F}[f]$ by the plane $\zeta = 0$, orthogonal to the z axis used for projection. There are numerous applications of this property in crystallography (Section 1.3.4.2.1.8) and in fibre diffraction (Section 1.3.4.5.1.3).

1.3.2.5.9. L^2 *aspects, Sobolev spaces*

The special properties of $\mathscr{F}$ in the space of square-integrable functions $L^2(\mathbb{R}^n)$, such as Parseval's identity, can be accommodated within distribution theory: if $u \in L^2(\mathbb{R}^n)$, then T_u is a tempered distribution in $\mathscr{S}'$ (the map $u \longmapsto T_u$ being continuous) and it can be shown that $S = \mathscr{F}[T_u]$ is of the form S_v, where $u = \mathscr{F}[u]$ is the Fourier transform of u in $L^2(\mathbb{R}^n)$. By Plancherel's theorem, $\|u\|_2 = \|v\|_2$.

This embedding of L^2 into $\mathscr{S}'$ can be used to derive the convolution theorem for L^2. If u and v are in $L^2(\mathbb{R}^n)$, then $u * v$ can be shown to be a bounded continuous function; thus $u * v$ is not in L^2, but it is in $\mathscr{S}'$, so that its Fourier transform is a distribution, and

$$\mathscr{F}[u * v] = \mathscr{F}[u] \times \mathscr{F}[v].$$

Spaces of tempered distributions related to $L^2(\mathbb{R}^n)$ can be defined as follows. For any real s, define the Sobolev space $H_s(\mathbb{R}^n)$ to consist of all tempered distributions $S \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$(1 + |\boldsymbol{\xi}|^2)^{s/2}\mathscr{F}[S]_{\boldsymbol{\xi}} \in L^2(\mathbb{R}^n).$$

These spaces play a fundamental role in the theory of partial differential equations, and in the mathematical theory of tomographic reconstruction – a subject not unrelated to the crystallographic phase problem (Natterer, 1986).

1.3.2.6. *Periodic distributions and Fourier series*

1.3.2.6.1. *Terminology*

Let $\mathbb{Z}^n$ be the subset of $\mathbb{R}^n$ consisting of those points with (signed) integer coordinates; it is an n-dimensional *lattice*, *i.e.* a free Abelian group on n generators. A particularly simple set of n generators is given by the standard basis of $\mathbb{R}^n$, and hence $\mathbb{Z}^n$ will be called the *standard lattice in* $\mathbb{R}^n$. Any other 'non-standard' n-dimensional lattice Λ in $\mathbb{R}^n$ is the image of this standard lattice by a general linear transformation.

If we identify any two points in $\mathbb{R}^n$ whose coordinates are congruent modulo $\mathbb{Z}^n$, *i.e.* differ by a vector in $\mathbb{Z}^n$, we obtain the *standard n-torus* $\mathbb{R}^n/\mathbb{Z}^n$. The latter may be viewed as $(\mathbb{R}/\mathbb{Z})^n$, *i.e.* as the Cartesian product of n circles. The same identification may be carried out modulo a non-standard lattice Λ, yielding a *non-*

references