If the basis $\mathbf{e}_i$ is orthonormal, $\Delta(g_{km})$ and V are equal to one, $\Delta(B_j)$ is equal to the volume V' of the cell built on the basis vectors $\mathbf{e}'_i$ and

$$\Delta(g'_{ij}) = V'^2.$$

This relation is actually general and one can remove the prime index:

$$\Delta(g_{ij}) = V^2. \qquad (1.1.2.21)$$

In the same way, we have for the corresponding reciprocal basis

$$\Delta(g^{ij}) = V^{*2},$$

where V^* is the volume of the reciprocal cell. Since the tables of the g_{ij}'s and of the g^{ij}'s are inverse, so are their determinants, and therefore the volumes of the unit cells of the direct and reciprocal spaces are also inverse, which is a very well known result in crystallography.

1.1.3. Mathematical notion of tensor

1.1.3.1. *Definition of a tensor*

For the mathematical definition of tensors, the reader may consult, for instance, Lichnerowicz (1947), Schwartz (1975) or Sands (1995).

1.1.3.1.1. *Linear forms*

A linear form in the space E_n is written

$$T(\mathbf{x}) = t_i x^i,$$

where $T(\mathbf{x})$ is independent of the chosen basis and the t_i's are the coordinates of T in the dual basis. Let us consider now a *bilinear* form in the product space $E_n \otimes F_p$ of two vector spaces with n and p dimensions, respectively:

$$T(\mathbf{x}, \mathbf{y}) = t_{ij} x^i y^j.$$

The np quantities t_{ij}'s are, *by definition*, the components of a tensor of rank 2 and the form $T(\mathbf{x}, \mathbf{y})$ is *invariant* if one changes the basis in the space $E_n \otimes F_p$. The tensor t_{ij} is said to be *twice covariant*. It is also possible to construct a bilinear form by replacing the spaces E_n and F_p by their respective conjugates E^n and F^p. Thus, one writes

$$T(\mathbf{x}, \mathbf{y}) = t_{ij} x^i y^j = t_i^{\,j} x^i y_j = t^i_{\,j} x_i y^j = t^{ij} x_i y_j,$$

where t^{ij} is the doubly contravariant form of the tensor, whereas $t_i^{\,j}$ and $t^i_{\,j}$ are mixed, once covariant and once contravariant.

We can generalize by defining in the same way tensors of rank 3 or higher by using trilinear or multilinear forms. A vector is a tensor of rank 1, and a scalar is a tensor of rank 0.

1.1.3.1.2. *Tensor product*

Let us consider two vector spaces, E_n with n dimensions and F_p with p dimensions, and let there be two linear forms, $T(\mathbf{x})$ in E_n and $S(\mathbf{y})$ in F_p. We shall associate with these forms a bilinear form called a *tensor product* which belongs to the product space with np dimensions, $E_n \otimes F_p$:

$$P(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}) \otimes S(\mathbf{y}).$$

This correspondence possesses the following properties:

(i) it is distributive from the right and from the left;

(ii) it is associative for multiplication by a scalar;

(iii) the tensor products of the vectors with a basis E_n and those with a basis F_p constitute a basis of the product space.

The analytical expression of the tensor product is then

$$\left.\begin{aligned} T(\mathbf{x}) &= t_i x^i \\ S(\mathbf{y}) &= s_j y^j \end{aligned}\right\} P(\mathbf{x}, \mathbf{y}) = p_{ij} x^i y^j = t_i x^i s_j y^j = t_i s_j x^i y^j.$$

One deduces from this that

$$p_{ij} = t_i s_j.$$

It is a tensor of rank 2. One can equally well envisage the tensor product of more than two spaces, for example, $E_n \otimes F_p \otimes G_q$ in npq dimensions. We shall limit ourselves in this study to the case of *affine* tensors, which are defined in a space constructed from the product of the space E_n with itself or with its conjugate E^n. Thus, a tensor product of rank 3 will have n^3 components. The tensor product can be generalized as the product of multilinear forms. One can write, for example,

$$\left.\begin{aligned} P(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= T(\mathbf{x}, \mathbf{y}) \otimes S(\mathbf{z}) \\ p^{\,j}_{i\,k} x^i y_j z^k &= t_i^{\,j} x^i y_j s_k z^k. \end{aligned}\right\} \qquad (1.1.3.1)$$

1.1.3.2. *Behaviour under a change of basis*

A multilinear form is, by definition, invariant under a change of basis. Let us consider, for example, the trilinear form (1.1.3.1). If we change the system of coordinates, the components of vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ become

$$x^i = B^i_\alpha x'^\alpha; \quad y_j = A^\beta_j y'_\beta; \quad z^k = B^k_\gamma z'^\gamma.$$

Let us put these expressions into the trilinear form (1.1.3.1):

$$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = p^{\,j}_{i\,k} B^i_\alpha x'^\alpha A^\beta_j y'_\beta B^k_\gamma z'^\gamma.$$

Now we can equally well make the components of the tensor appear in the new basis:

$$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = p'^{\,\beta}_{\alpha\,\gamma} x'^\alpha y'_\beta z'^\gamma.$$

As the decomposition is unique, one obtains

$$p'^{\,\beta}_{\alpha\,\gamma} = p^{\,j}_{i\,k} B^i_\alpha A^\beta_j B^k_\gamma. \qquad (1.1.3.2)$$

One thus deduces the rule for transforming the components of a tensor q times covariant and r times contravariant: they transform like the product of q covariant components and r contravariant components.

This transformation rule can be taken inversely as the definition of the components of a tensor of rank $n = q + r$.

Example. The operator O representing a symmetry operation has the character of a tensor. In fact, under a change of basis, O transforms into O':

$$O' = AOA^{-1}$$

so that

$$O'^i_j = A^i_k O^k_l (A^{-1})^l_j.$$

Now the matrices A and B are inverses of one another:

$$O'^i_j = A^i_k O^k_l B^l_j.$$

The symmetry operator is a tensor of rank 2, once covariant and once contravariant.

1.1.3.3. *Operations on tensors*

1.1.3.3.1. *Addition*

It is necessary that the tensors are of the same nature (same rank and same variance).