1.1. Introduction to the properties of tensors

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1.1.1. The matrix of physical properties

1.1.1.1. *Notion of extensive and intensive quantities*

Physical laws express in general the response of a medium to a certain influence. Most physical properties may therefore be defined by a relation coupling two or more measurable quantities. For instance, the specific heat characterizes the relation between a variation of temperature and a variation of entropy at a given temperature in a given medium, the dielectric susceptibility the relation between electric field and electric polarization, the elastic constants the relation between an applied stress and the resulting strain *etc*. These relations are between quantities of the same nature: thermal, electrical and mechanical, respectively. But there are also cross effects, for instance:

(*a*) *thermal expansion* and *piezocalorific effect*: mechanical reaction to a thermal impetus or the reverse;

(*b*) *pyroelectricity* and *electrocalorific effect*: electrical response to a thermal impetus or the reverse;

(*c*) *piezoelectricity* and *electrostriction*: electric response to a mechanical impetus;

(*d*) *piezomagnetism* and *magnetostriction*: magnetic response to a mechanical impetus;

(*e*) *photoelasticity*: birefringence produced by stress;

(*f*) *acousto-optic effect*: birefringence produced by an acoustic wave;

(*g*) *electro-optic effect*: birefringence produced by an electric field;

(*h*) *magneto-optic effect*: appearance of a rotatory polarization under the influence of a magnetic field.

The physical quantities that are involved in these relations can be divided into two categories:

(i) *extensive quantities*, which are proportional to the volume of matter or to the mass, that is to the number of molecules in the medium, for instance entropy, energy, quantity of electricity *etc.* One uses frequently specific extensive parameters, which are given per unit mass or per unit volume, such as the specific mass, the electric polarization (dipole moment per unit volume) *etc.*

(ii) *intensive parameters*, quantities whose product with an extensive quantity is homogeneous to an energy. For instance, volume is an extensive quantity; the energy stored by a gas undergoing a change of volume $\mathrm{d}V$ under pressure p is $p\,\mathrm{d}V$. Pressure is therefore the intensive parameter associated with volume. Table 1.1.1.1 gives examples of extensive quantities and of the related intensive parameters.

1.1.1.2. *Notion of tensor in physics*

Each of the quantities mentioned in the preceding section is represented by a mathematical expression. Some are direction independent and are represented by *scalars*: specific mass, specific heat, volume, pressure, entropy, temperature, quantity of electricity, electric potential. Others are direction dependent and are represented by *vectors*: force, electric field, electric displacement, the gradient of a scalar quantity. Still others cannot be represented by scalars or vectors and are represented by more complicated mathematical expressions. Magnetic quantities are represented by *axial* vectors (or pseudovectors), which are a particular kind of tensor (see Section 1.1.4.5.3). A few examples will show the necessity of using tensors in physics and Section 1.1.3 will present elementary mathematical properties of tensors.

(i) *Thermal expansion*. In an isotropic medium, thermal expansion is represented by a single number, a scalar, but this is not the case in an anisotropic medium: a sphere cut in an anisotropic medium becomes an ellipsoid when the temperature is varied and thermal expansion can no longer be represented by a single number. It is actually represented by a tensor of rank 2.

(ii) *Dielectric constant*. In an isotropic medium of a perfect dielectric we can write, in SI units,

$$\begin{aligned}\mathbf{P} &= \varepsilon_0 \chi_e \mathbf{E}\\ \mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E},\end{aligned}$$

where $\mathbf{P}$ is the electric polarization (= dipole moment per unit volume), ε_0 the permittivity of vacuum, χ_e the dielectric susceptibility, $\mathbf{D}$ the electric displacement and ε the dielectric constant, also called dielectric permittivity. These expressions indicate that the electric field, on the one hand, and polarization and displacement, on the other hand, are linearly related. In the general case of an anisotropic medium, this is no longer true and one must write expressions indicating that the components of the displacement are linearly related to the components of the field:

$$\left\{\begin{aligned} D^1 &= \varepsilon_1^1 E^1 + \varepsilon_1^2 E^2 + \varepsilon_1^3 E\\ D^2 &= \varepsilon_1^2 E^1 + \varepsilon_2^2 E^2 + \varepsilon_2^3 E\\ D^3 &= \varepsilon_1^3 E^1 + \varepsilon_2^3 E^2 + \varepsilon_3^3 E.\end{aligned}\right. \qquad (1.1.1.1)$$

The dielectric constant is now characterized by a set of nine components ε_i^j; they are the components of a tensor of rank 2. It will be seen in Section 1.1.4.5.2.1 that this tensor is symmetric ($\varepsilon_i^j = \varepsilon_j^i$) and that the number of independent components is equal to six.

(iii) *Stressed rod (Hooke's law)*. If one pulls a rod of length ℓ and cross section $\mathcal{A}$ with a force F, its length is increased by a quantity $\Delta\ell$ given by $\Delta\ell/\ell = (1/E)F/\mathcal{A}$, where E is Young's modulus, or elastic stiffness (see Section 1.3.3.1). But, at the same time, the radius, r, decreases by Δr given by $\Delta r/r = -(\nu/E)F/\mathcal{A}$, where ν is Poisson's ratio (Section 1.3.3.4.3). It can be seen that a scalar is not sufficient to describe the elastic deformation of a material, even if it is isotropic. The number of independent components depends on the symmetry of the medium and it will be seen that they are the components of a tensor of rank 4. It was precisely to describe the properties of elasticity by a mathematical expression that the notion of a tensor was introduced in physics by W. Voigt in the 19th century (Voigt, 1910) and by L. Brillouin in the first half of the 20th century (Brillouin, 1949).

Table 1.1.1.1. *Extensive quantities and associated intensive parameters*

The last four lines of the table refer to properties that are time dependent.

Extensive quantities	Intensive parameters
Volume	Pressure
Strain	Stress
Displacement	Force
Entropy	Temperature
Quantity of electricity	Electric potential
Electric polarization	Electric field
Electric displacement	Electric field
Magnetization	Magnetic field
Magnetic induction	Magnetic field
Reaction rate	Chemical potential
Heat flow	Temperature gradient
Diffusion of matter	Concentration gradient
Electric current	Potential gradient

references