1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

$$
\left.\begin{array}{lll}
 & & \text{compliances} \\
i = j; & k = l & s_{\alpha\beta} = siill \\
i = j; & k \neq l & s_{\alpha\beta} = siill + siilk \\
i \neq j; & k = l & s_{\alpha\beta} = sijll + sjill \\
i \neq j; & k \neq l & s_{\alpha\beta} = sijkl + sjikl + sijlk + sjilk
\end{array}\right\} \quad
\begin{array}{l}
\text{stiffnesses} \\
c_{\alpha\beta} = cijkl.
\end{array}
$$

$$(1.3.3.6)$$

Using these notations and Voigt's notations for the strain and stress tensors, equations (1.3.3.2) become:

$$
\begin{aligned}
S_{\alpha\beta} &= s_{\alpha\beta}T_\beta \\
T_{\alpha\beta} &= c_{\alpha\beta}S_\beta.
\end{aligned}
$$

$$(1.3.3.7)$$

For instance, the first of these two relations is written, once developed, as

$$
\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{pmatrix} =
\begin{pmatrix}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\
s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\
s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\
s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\
s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\
s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66}
\end{pmatrix}
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix}.
$$

$$(1.3.3.8)$$

Matrices $s_{\alpha\beta}$ and $c_{\alpha\beta}$ are the inverse of each another. It is important to note that they are matrices and not tensors. One cannot apply to them the usual rules of transformation under a change of base since they are only valid for the components of a tensor. In cases where the coordinate system is changed, it is necessary to use the components s_{ijkl} and c_{ijkl} or to establish the rules of transformation for $s_{\alpha\beta}$ and $c_{\alpha\beta}$.

(iii) *Symmetry of the crystal*: the reduction of the number of independent components of the matrices $s_{\alpha\beta}$ and $c_{\alpha\beta}$ was discussed in Section 1.1.4.10.6. As a summary, Table 1.3.3.1 gives the number of independent components for each Laue class.

(iv) *Cauchy relation*: the form that the strain energy of a discrete-particle structure takes when its potential energy depends only on the magnitude of the distance separating pairs of particles is called the central or point-to-point force law. It is shown that, when the cohesive forces have this special form, structures that are such that every atom is situated at a centre of symmetry have elastic constants that are totally symmetric in their four indices. Thus

$$c_{ijkl} = c_{ikjl} = c_{iljk}$$

in addition to the symmetries required by the previous considerations. This additional symmetry implies the following relations between the two-index components:

$$
\begin{array}{ll}
c_{23} = c_{44}; & c_{14} = c_{56} \\
c_{13} = c_{55}; & c_{25} = c_{46} \\
c_{12} = c_{66}; & c_{36} = c_{45}.
\end{array}
$$

Table 1.3.3.1. *Number of independent components of the elastic compliances and stiffnesses for each Laue class*

Laue class	No. of independent components
$\bar{1}, 1$	21
$2/m, 2, m$	13
$mmm, 222, 2mm$	9
$\bar{3}, 3$	7
$\bar{3}m, 32, 3m$	6
$4/m, 4, \bar{4}$	7
$4/mmm, 422, \bar{4}2m, 42m$	6
$6/m, 6, \bar{6}$	5
$6/mmm, 622, \bar{6}2m, 62m$	5
$m\bar{3}, 23$	3
$m\bar{3}m, 432, \bar{4}32$	3
$\infty A_\infty/MC, \infty A_\infty$	2

These are known as the Cauchy relations. The further symmetry implied by the Cauchy relations reduces the maximum number of independent constants from 21 to 15. In crystal structures for which they might be valid, the extent to which these relations are fulfilled is often used to assess the validity of the assumption of a central-force law. It is important to apply such a test only to structures having the necessary symmetry properties for Cauchy relations to hold. For instance, f.c.c and b.c.c. metals have the required symmetry (an inversion centre at each atomic site), while diamond, silicon and germanium do not. Consequently, any apparent fulfilment of the Cauchy relation $c_{12} = c_{66}$ for a diamond-type structure offers no ground for conclusions about the nature of the force field.

1.3.3.2.3. *Passage from elastic compliances $s_{\alpha\beta}$ to elastic stiffnesses $c_{\alpha\beta}$*

We have noted already that the matrix $c_{\alpha\beta}$ is the inverse of the matrix $s_{\alpha\beta}$. These matrices can be written for cubic and isotropic materials as follows:

$$
s_{\alpha\beta} =
\begin{pmatrix}
s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\
s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\
s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & s_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & s_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & s_{44}
\end{pmatrix}
$$

$$
c_{\alpha\beta} =
\begin{pmatrix}
c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\
c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{44}
\end{pmatrix},
$$

where we have, for isotropic materials,

$$
\begin{aligned}
s_{44} &= 2(s_{11} - s_{12}) \\
c_{44} &= \tfrac{1}{2}(c_{11} - c_{12}).
\end{aligned}
$$

$$(1.3.3.9)$$

We easily find that

$$
\begin{aligned}
s_{11} &= \frac{c_{11} + c_{12}}{(c_{11} - c_{12})(c_{11} + 2c_{12})}; & c_{11} &= \frac{s_{11} + s_{12}}{(s_{11} - s_{12})(s_{11} + 2s_{12})}; \\
s_{12} &= \frac{-c_{12}}{(c_{11} - c_{12})(c_{11} + 2c_{12})}; & c_{12} &= \frac{-s_{12}}{(s_{11} - s_{12})(s_{11} + 2s_{12})}; \\
s_{44} &= \frac{1}{c_{44}}; & c_{44} &= \frac{1}{s_{44}}.
\end{aligned}
$$

The coefficient c_{44} is sometimes called the *rigidity modulus*.

1.3.3.3. *Elastic strain energy*

Expression (1.3.2.7) of the strain energy stored per unit volume in a medium for a small deformation can be integrated when the medium is strained under a stress T_{ij} according to linear elasticity. Applying relation (1.3.3.2), one gets for the density of strain energy

$$W = \tfrac{1}{2}T_{ij}S_{ij} = \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl}.$$

$$(1.3.3.10)$$

1.3.3.4. *Particular elastic constants*

1.3.3.4.1. *Volume compressibility*

Let us apply a hydrostatic pressure (Section 1.3.2.5.2). The medium undergoes a relative variation of volume $\Delta V/V = S_1 + S_2 + S_3$ (the cubic dilatation, Section 1.3.1.3.2). If one

references