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SUBPERIODIC GROUPS

Edited by
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Dedicated to

Mary,

whose memory I will cherish forever.

V. KOPSKÝ

Tikva (זכרונה לברכה) and Professor W. Opechowski,
may their memories be blessed.

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Contents

	PAGE
Preface	ix
PART 1. SUBPERIODIC GROUP TABLES: FRIEZE-GROUP, ROD-GROUP AND LAYER-GROUP TYPES	1
1.1. Symbols and terms used in Parts 1–4	2
1.2. Guide to the use of the subperiodic group tables	5
1.2.1. Classification of subperiodic groups	5
1.2.2. Contents and arrangement of the tables	7
1.2.3. Headline	7
1.2.4. International (Hermann–Mauguin) symbols for subperiodic groups	7
1.2.5. Patterson symmetry	8
1.2.6. Subperiodic group diagrams	8
1.2.7. Origin	14
1.2.8. Asymmetric unit	14
1.2.9. Symmetry operations	15
1.2.10. Generators	16
1.2.11. Positions	16
1.2.12. Oriented site-symmetry symbols	16
1.2.13. Reflection conditions	17
1.2.14. Symmetry of special projections	17
1.2.15. Maximal subgroups and minimal supergroups	19
1.2.16. Nomenclature	21
1.2.17. Symbols	22
References	24
PART 2. THE 7 FRIEZE GROUPS	31
PART 3. THE 75 ROD GROUPS	39
PART 4. THE 80 LAYER GROUPS	221
PART 5. SCANNING OF SPACE GROUPS	393
5.1. Symbols used in Parts 5 and 6	394
5.2. Guide to the use of the scanning tables	395
5.2.1. Introduction	395
5.2.2. The basic concepts of the scanning	396
5.2.3. The contents and arrangement of the scanning tables	401
5.2.4. Guidelines for individual systems	405
5.2.5. Applications	414
References	419
PART 6. THE SCANNING TABLES	421
Author index	565
Subject index	566

Preface

BY V. KOPSKÝ AND D. B. LITVIN

This volume is divided into two sections. The first, covered in Parts 1–4, can be considered as an extension of Volume A: *Space-Group Symmetry*, in this series of *International Tables for Crystallography*. As Volume A treats one-, two-, and three-dimensional space groups, this Volume treats the two- and three-dimensional subperiodic groups. That is, it treats the frieze groups, two-dimensional groups with one-dimensional translations, the rod groups, three-dimensional groups with one-dimensional translations, and layer groups, three-dimensional groups with two-dimensional translations. A reader familiar with Volume A should readily recognize the format and content of the tables of Parts 1–4 of this volume. The information presented for the subperiodic groups is in the same format and consists of the same content as that provided in Volume A for space groups.

A relationship between space and subperiodic groups is considered in Parts 5 and 6: given a crystal of a specific space-group symmetry and a plane transecting the crystal, one can enquire as to what is the layer subgroup of the space group which leaves this plane invariant. The physical motivation for answering this question is discussed in Chapter 5.2. This is followed by the ‘Scanning Tables’ in which the layer symmetries of ‘sectional’ planes are tabulated for all crystallographic orientations and for all positions (locations) of these planes. These tables also contain explicitly the orbits of these planes and implicitly, *via* the so-called ‘scanning groups’, information about the rod symmetries of straight lines which penetrate through the crystal.

A new feature of this second edition is the addition of Seitz notation. In the symmetry-operations section of each table in Parts 2–4 the Seitz notation for each symmetry operation is given below its geometric notation. Minor additions to the text and tables, and corrections of a few typographical errors, have been made.

The history of this work dates back to 1972 when one of us (DBL) was asked by a fellow post doc, John Berlinsky, if there existed International-like tables to classify arrays of hydrogen molecules on a surface with the molecules not constrained to be ‘in-plane’. Tables for the layer groups were subsequently derived in the content and format of the *International Tables for X-ray Crystallography*, Volume 1 (1952). It was later pointed out by a referee of *Acta Crystallographica* that such tables had already been published by E. Wood in 1964. Work on these tables remained dormant until 1983 with the publication of Volume A of *International Tables for Crystallography*, and the extensive addition of new features in the description of each space group. Work began then on including these new features into tables for the layer groups.

During this same time one of us (VK) was asked by Dr V. Janovec to investigate the group-theoretical aspects of the analysis of domain walls and twin boundaries. Thus, work began on the relationships between space groups and subperiodic groups and standards for the subperiodic groups.

It is our subsequent collaboration which has led to the material presented in this volume. In the many decisions concerning the choice of symbols, origins and settings for the subperiodic groups, the final choices were made based on relationships between space groups and subperiodic groups. While these relationships are not all explicitly given here, they have been implicitly used.

As with any major work such as this, there are those to whom we must give our thanks: Dr E. Woods is thanked for her encouragement during the initial stage of this work. Dr Th. Hahn has provided advice, comments and encouragement dating back to 1983. Constructive feedback on reading parts of this work were received from Drs Th. Hahn, H. Wondratschek and V. Janovec. The drawings in Parts 1–4 of this volume were done by Steven Erb, a Mechanical Engineering Technology student at the Berks Campus of the Pennsylvania State University. The drawings in Parts 5 and 6 were done by V. Kopský Jr, a biology student at Charles University. We also thank M. I. Aroyo, P. Konstantinov, E. Kroumova and M. Gateshki for converting the computer files of Parts 2, 3 and 4 from WordPerfect to L^AT_EX format.

The financial support received from various organizations during which work was performed leading to and for this volume from a National Academy of Science–Czechoslovak Academy of Science Exchange Program (1984), the United States National Science Foundation (INT-8922251), the International Union of Crystallography and the Pennsylvania State University is gratefully acknowledged by us. In addition, for their major additional support DBL thanks the United States National Science Foundation (DMR-8406196, DMR-9100418, DMR-9305825 and DMR-9510335) and VK the University of the South Pacific (Fiji) (Research Committee Grant 070-91111), under which a major portion of this work was completed in an idyllic setting, and the Grant Agency of the Czech Republic (GA CR 202/96/1614).

As to the dedication, we would like to point out, to quell any rumours to the contrary, that Mary and Tikva (זכרונה לברכה) are our respective wives. Their unending patience and constant encouragement are indeed due recognition. The parenthetical Hebrew means ‘may her memory be blessed’, and Professor W. Opechowski is included as DBL’s scientific ‘father’.

SAMPLE PAGES

1.2. Guide to the use of the subperiodic group tables

This present volume is, in part, an extension of *International Tables for Crystallography*, Volume A, *Space-Group Symmetry* (IT A, 2005). Symmetry tables are given in IT A for the 230 three-dimensional crystallographic space-group types (space groups) and the 17 two-dimensional crystallographic space-group types (plane groups). We give in the following three parts of this volume analogous symmetry tables for the two-dimensional and three-dimensional subperiodic group types: the seven crystallographic *frieze-group* types (two-dimensional groups with one-dimensional translations) in Part 2; the 75 crystallographic *rod-group* types (three-dimensional groups with one-dimensional translations) in Part 3; and the 80 crystallographic *layer-group* types (three-dimensional groups with two-dimensional translations) in Part 4. This chapter forms a guide to the entries of the subperiodic group tables given in Parts 2–4.

1.2.1. Classification of subperiodic groups

Subperiodic groups can be classified in ways analogous to the space groups. For the mathematical definitions of these classifications and their use for space groups, see Chapter 8.2 of IT A (2005). Here we shall limit ourselves to those classifications which are explicitly used in the symmetry tables of the subperiodic groups.

1.2.1.1. Subperiodic group types

The subperiodic groups are classified into *affine subperiodic group types*, *i.e.* affine equivalence classes of subperiodic groups. There are 80 affine layer-group types and seven affine frieze-group types. There are 67 crystallographic and an infinity of noncrystallographic (Opechowski, 1986) affine rod-group types. We shall consider here only rod groups of the 67 crystallographic rod-group types. We shall refer here to these crystallographic affine rod-group types simply as affine rod-group types and to the crystallographic rod groups belonging to these types simply as rod groups.

The subperiodic groups are also classified into *proper affine subperiodic group types*, *i.e.* proper affine classes of subperiodic groups. For layer and frieze groups, the two classifications are identical. For rod groups, each of eight affine rod-group types splits into a pair of *enantiomorphic crystallographic rod-group types*. Consequently, there are 75 proper affine rod-group types. The eight pairs of enantiomorphic rod-group types are $\mu 4_1$ (R24), $\mu 4_3$ (R26); $\mu 4_1 22$ (R31), $\mu 4_3 22$ (R33); $\mu 3_1$ (R43), $\mu 3_2$ (R44); $\mu 3_1 12$ (R47), $\mu 3_2 12$ (R48); $\mu 6_1$ (R54), $\mu 6_5$ (R58); $\mu 6_2$ (R55), $\mu 6_4$ (R57); $\mu 6_1 22$ (R63), $\mu 6_5 22$ (R67); and $\mu 6_2 22$ (R64), $\mu 6_4 22$ (R66). (Each subperiodic group is given in the text by its Hermann–Mauguin symbol followed in parenthesis by a letter L, R or F to denote it, respectively, as a layer, rod or frieze group, and its sequential numbering from Parts 2, 3 or 4.) We shall refer to the proper affine subperiodic group types simply as subperiodic group types.

1.2.1.2. Other classifications

There are 27 geometric crystal classes of layer groups and rod groups, and four geometric crystal classes of frieze groups. These

are listed, for layer groups, in the fourth column of Table 1.2.1.1, and for the rod and frieze groups in the second columns of Tables 1.2.1.2 and 1.2.1.3, respectively.

We further classify subperiodic groups according to the following classifications of the subperiodic group's point group and lattice group. These classifications are introduced to emphasize the relationships between subperiodic groups and space groups:

(1) The point group of a layer or rod group is three-dimensional and corresponds to a point group of a three-dimensional space group. The point groups of three-dimensional space groups are classified into the triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal and cubic crystal systems. We shall use this classification also for subperiodic groups. Consequently, the three-dimensional subperiodic groups are classified, see the third column of Table 1.2.1.1 and the first column of Table 1.2.1.2, into the triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal crystal systems. The cubic crystal system does not arise for three-dimensional subperiodic groups. Two-dimensional subperiodic groups, *i.e.* frieze groups, are analogously classified, see the first column of Table 1.2.1.3, into the oblique and rectangular crystal systems.

(2) The two-dimensional lattice of a layer group is also a two-dimensional lattice of a plane group. The lattices of plane groups are classified, according to *Bravais (flock) systems*, see IT A (2005), into the oblique, rectangular, square and hexagonal Bravais systems. We shall also use this classification for layer groups, see the first column in Table 1.2.1.1. For rod and frieze groups no lattice classification is used, as all one-dimensional lattices form a single Bravais system.

A subdivision of the monoclinic rod-group category is made into monoclinic/inclined and monoclinic/orthogonal. Two different coordinate systems, see Table 1.2.1.2, are used for the rod groups of these two subdivisions of the monoclinic crystal system. These two coordinate systems differ in the orientation of the plane containing the non-lattice basis vectors relative to the lattice vectors. For the monoclinic/inclined subdivision, the plane containing the non-lattice basis vectors is, see Fig. 1.2.1.1, *inclined* with respect to the lattice basis vector. For the monoclinic/orthogonal subdivision, the plane is, see Fig. 1.2.1.2, *orthogonal*.

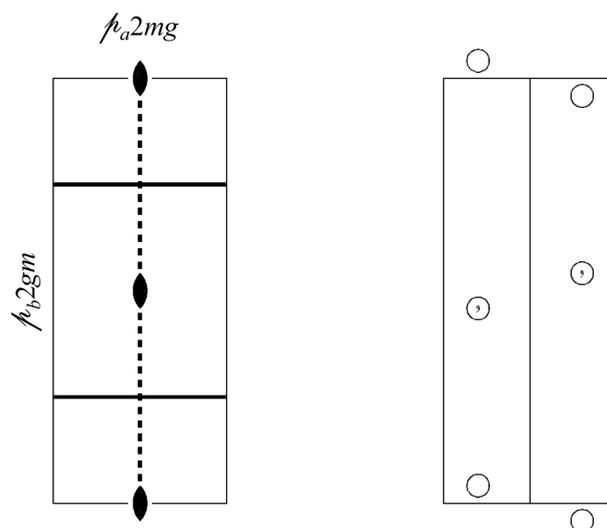
1.2.1.2.1. Conventional coordinate systems

The subperiodic groups are described by means of a *crystallographic coordinate system* consisting of a *crystallographic origin*, denoted by *O*, and a *crystallographic basis*. The basis vectors for the three-dimensional layer groups and rod groups are labelled **a**, **b** and **c**. The basis vectors for the two-dimensional frieze groups are labelled **a** and **b**. Unlike space groups, not all basis vectors of the crystallographic basis are lattice vectors. Like space groups, the crystallographic coordinate system is used to define the symmetry operations (see Section 1.2.9) and the Wyckoff positions (see Section 1.2.11). The symmetry operations are defined with respect to the directions of both lattice and non-lattice basis vectors. A Wyckoff position, denoted by a coordinate triplet (*x*, *y*, *z*) for the three-dimensional layer and rod groups, is

$\mu 2mg$ $2mm$

Rectangular

No. 7

 $\mu 2mg$ Patterson symmetry $\mu 2mm$ Origin at $21g$ Asymmetric unit $0 \leq x \leq \frac{1}{4}$

Symmetry operations

(1) 1 (2) $2 \ 0,0$ (3) $m \ \frac{1}{4},y$ (4) $g \ x,0$
 (1|0,0) (2|0,0) ($m_x|\frac{1}{2},0$) ($m_y|\frac{1}{2},0$)

Generators selected (1); $t(1)$; (2); (3)

Positions

Multiplicity,
Wyckoff letter,
Site symmetry

Coordinates

Reflection conditions

4 $c \ 1$ (1) x,y (2) \bar{x},\bar{y} (3) $\bar{x}+\frac{1}{2},y$ (4) $x+\frac{1}{2},\bar{y}$

General:

 $h : h = 2n$ 2 $b \ .m.$ $\frac{1}{4},y$ $\frac{3}{4},\bar{y}$ 2 $a \ 2..$ $0,0$ $\frac{1}{2},0$

Special: no extra conditions

Symmetry of special projections

Along $[10] \ m$ Along $[01] \ \mu m$ Origin at $x,0$ $\mathbf{a}' = \frac{1}{2}\mathbf{a}$ Origin at $0,y$

Maximal non-isotypic subgroups

I $[2] \mu 11g$ (5) 1; 4 $[2] \mu 1m1$ (3) 1; 3 $[2] \mu 211$ (2) 1; 2

IIa none

IIb none

Maximal isotypic subgroups of lowest index

IIc $[3] \mu 2mg$ ($\mathbf{a}' = 3\mathbf{a}$) (7)

Minimal non-isotypic supergroups

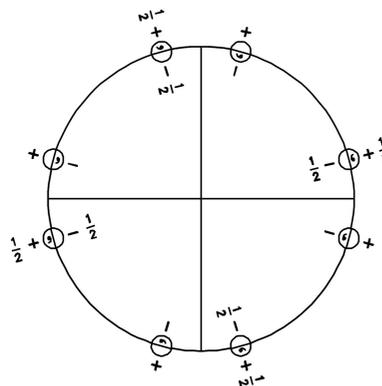
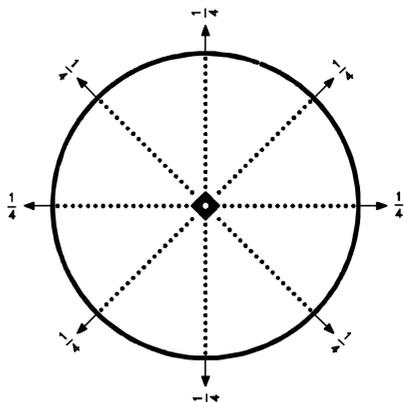
I none

II $[2] \mu 2mm$ ($\mathbf{a}' = \frac{1}{2}\mathbf{a}$) (6)

$\mu 4/mcc$ $4/mmm$

Tetragonal

No. 40

 $\mu 4/m2/c2/c$ Patterson symmetry $\mu 4/mmm$ **Origin** at centre ($4/m$) at $4/mcc$ **Asymmetric unit** $0 \leq x; 0 \leq y; 0 \leq z \leq \frac{1}{4}$ **Symmetry operations**

- | | | | |
|---|---|--|--|
| (1) 1
(1 0,0,0) | (2) 2 0,0,z
(2 _z 0,0,0) | (3) 4 ⁺ 0,0,z
(4 _z ⁺ 0,0,0) | (4) 4 ⁻ 0,0,z
(4 _z ⁻ 0,0,0) |
| (5) 2 0,y, $\frac{1}{4}$
(2 _y 0,0, $\frac{1}{2}$) | (6) 2 x,0, $\frac{1}{4}$
(2 _x 0,0, $\frac{1}{2}$) | (7) 2 x,x, $\frac{1}{4}$
(2 _{xy} 0,0, $\frac{1}{2}$) | (8) 2 x, \bar{x} , $\frac{1}{4}$
(2 _{xy} 0,0, $\frac{1}{2}$) |
| (9) $\bar{1}$ 0,0,0
($\bar{1}$ 0,0,0) | (10) m x,y,0
(m _z 0,0,0) | (11) $\bar{4}^+$ 0,0,z; 0,0,0
($\bar{4}_z^+$ 0,0,0) | (12) $\bar{4}^-$ 0,0,z; 0,0,0
($\bar{4}_z^-$ 0,0,0) |
| (13) c x,0,z
(m _y 0,0, $\frac{1}{2}$) | (14) c 0,y,z
(m _x 0,0, $\frac{1}{2}$) | (15) c x, \bar{x} ,z
(m _{xy} 0,0, $\frac{1}{2}$) | (16) c x,x,z
(m _{xy} 0,0, $\frac{1}{2}$) |

Generators selected (1); $t(0,0,1)$; (2); (3); (5); (9)

Positions

Multiplicity, Wyckoff letter, Site symmetry	Coordinates	Reflection conditions
16 <i>g</i> 1	(1) x, y, z (2) \bar{x}, \bar{y}, z (3) \bar{y}, x, z (4) y, \bar{x}, z (5) $\bar{x}, y, \bar{z} + \frac{1}{2}$ (6) $x, \bar{y}, \bar{z} + \frac{1}{2}$ (7) $y, x, \bar{z} + \frac{1}{2}$ (8) $\bar{y}, \bar{x}, \bar{z} + \frac{1}{2}$ (9) $\bar{x}, \bar{y}, \bar{z}$ (10) x, y, \bar{z} (11) y, \bar{x}, \bar{z} (12) \bar{y}, x, \bar{z} (13) $x, \bar{y}, z + \frac{1}{2}$ (14) $\bar{x}, y, z + \frac{1}{2}$ (15) $\bar{y}, \bar{x}, z + \frac{1}{2}$ (16) $y, x, z + \frac{1}{2}$	General: $l : l = 2n$ Special: no extra conditions
8 <i>f</i> $m..$	$x, y, 0$ $\bar{x}, y, \frac{1}{2}$ $\bar{y}, x, 0$ $y, x, \frac{1}{2}$	
8 <i>e</i> $.2.$	$x, 0, \frac{1}{4}$ $\bar{x}, 0, \frac{3}{4}$ $0, x, \frac{1}{4}$ $0, \bar{x}, \frac{3}{4}$	
8 <i>d</i> $..2$	$x, x, \frac{1}{4}$ $\bar{x}, \bar{x}, \frac{3}{4}$ $\bar{x}, x, \frac{1}{4}$ $x, \bar{x}, \frac{3}{4}$	
4 <i>c</i> $4..$	$0, 0, z$ $0, 0, \bar{z} + \frac{1}{2}$ $0, 0, \bar{z}$ $0, 0, z + \frac{1}{2}$	
2 <i>b</i> $4/m..$	$0, 0, 0$ $0, 0, \frac{1}{2}$	
2 <i>a</i> 422	$0, 0, \frac{1}{4}$ $0, 0, \frac{3}{4}$	

Symmetry of special projections

Along [001] $4mm$

Along [100] $\mu 2mm$

Along [110] $\mu 2mm$

Origin at $0, 0, z$

$\mathbf{a}' = \frac{1}{2}\mathbf{c}$
Origin at $x, 0, 0$

$\mathbf{a}' = \frac{1}{2}\mathbf{c}$
Origin at $x, x, 0$

Maximal non-isotypic non-enantiomorphic subgroups

I	$[2]\mu 42c$ (38)	1; 2; 5; 6; 11; 12; 15; 16
	$[2]\mu 4c2$ ($\mu 42c$, 38)	1; 2; 7; 8; 11; 12; 13; 14
	$[2]\mu 4cc$ (36)	1; 2; 3; 4; 13; 14; 15; 16
	$[2]\mu 422$ (30)	1; 2; 3; 4; 5; 6; 7; 8
	$[2]\mu 4/m11$ ($\mu 4/m$, 28)	1; 2; 3; 4; 9; 10; 11; 12
	$[2]\mu 2/m2/c1$ (μccm , 21)	1; 2; 5; 6; 9; 10; 13; 14
	$[2]\mu 2/m12/c$ (μccm , 21)	1; 2; 7; 8; 9; 10; 15; 16

IIa none

IIb none

Maximal isotypic subgroups and enantiomorphic subgroups of lowest index

IIc $[3]\mu 4/mcc$ ($\mathbf{c}' = 3\mathbf{c}$) (40)

Minimal non-isotypic non-enantiomorphic supergroups

I none

II $[2]\mu 4/mmm$ ($\mathbf{c}' = \frac{1}{2}\mathbf{c}$) (39)

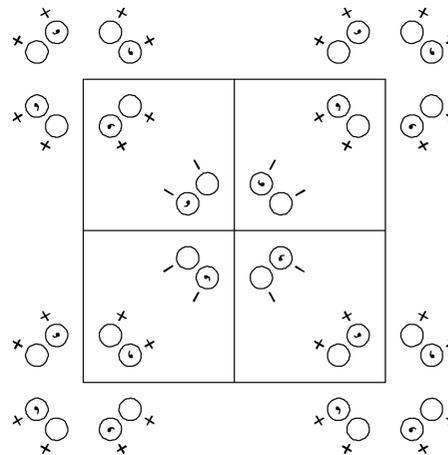
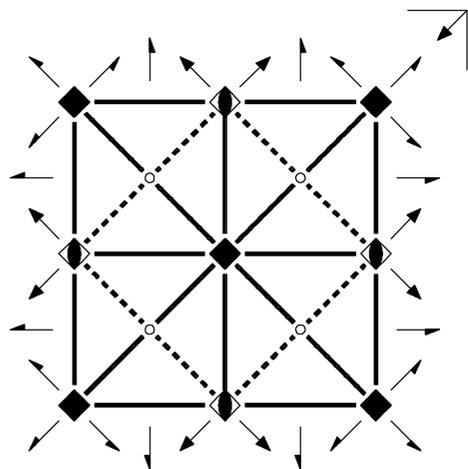
$p4/nmm$ $4/mmm$

Tetragonal/Square

No. 64

 $p4/n2_1/m2/m$ Patterson symmetry $p4/mmm$

ORIGIN CHOICE 1

**Origin** on $4mm$ at $-\frac{1}{4}, -\frac{1}{4}, 0$ from centre ($2/m$)**Asymmetric unit** $0 \leq x \leq \frac{1}{2}; 0 \leq y \leq \frac{1}{2}; y \leq \frac{1}{2} - x; 0 \leq z$ **Symmetry operations**

- | | | | |
|---|---|--|---|
| (1) 1
(1 0,0,0) | (2) 2 0,0,z
(2 _z 0,0,0) | (3) 4 ⁺ 0,0,z
(4 _z ⁺ 0,0,0) | (4) 4 ⁻ 0,0,z
(4 _z ⁻ 0,0,0) |
| (5) 2($\frac{1}{2}, \frac{1}{2}, 0$) 0,y,0
(2 _y \frac{1}{2}, \frac{1}{2}, 0) | (6) 2($\frac{1}{2}, \frac{1}{2}, 0$) x,0,0
(2 _x \frac{1}{2}, \frac{1}{2}, 0) | (7) 2($\frac{1}{2}, \frac{1}{2}, 0$) x,x,0
(2 _{xy} \frac{1}{2}, \frac{1}{2}, 0) | (8) 2($\frac{1}{2}, \frac{1}{2}, 0$) x, \bar{x} ,0
(2 _{$\bar{y}x$} \frac{1}{2}, \frac{1}{2}, 0) |
| (9) $\bar{1}$ $\frac{1}{4}, \frac{1}{4}, 0$
($\bar{1}$ \frac{1}{2}, \frac{1}{2}, 0) | (10) n ($\frac{1}{2}, \frac{1}{2}, 0$) x,y,0
(m_z \frac{1}{2}, \frac{1}{2}, 0) | (11) $\bar{4}^+$ $\frac{1}{2}, 0, z; \frac{1}{2}, 0, 0$
($\bar{4}_z^+$ \frac{1}{2}, \frac{1}{2}, 0) | (12) $\bar{4}^-$ $\frac{1}{2}, 0, z; \frac{1}{2}, 0, 0$
($\bar{4}_z^-$ \frac{1}{2}, \frac{1}{2}, 0) |
| (13) m x,0,z
(m_y 0,0,0) | (14) m 0,y,z
(m_x 0,0,0) | (15) m x, \bar{x} ,z
(m_{xy} 0,0,0) | (16) m x,x,z
($m_{\bar{y}x}$ 0,0,0) |

Generators selected (1); $t(1,0,0)$; $t(0,1,0)$; (2); (3); (5); (9)

Positions

Multiplicity, Wyckoff letter, Site symmetry	Coordinates				Reflection conditions
16 h 1	(1) x, y, z (5) $\bar{x} + \frac{1}{2}, y + \frac{1}{2}, \bar{z}$ (9) $\bar{x} + \frac{1}{2}, \bar{y} + \frac{1}{2}, \bar{z}$ (13) x, \bar{y}, z	(2) \bar{x}, \bar{y}, z (6) $x + \frac{1}{2}, \bar{y} + \frac{1}{2}, \bar{z}$ (10) $x + \frac{1}{2}, y + \frac{1}{2}, \bar{z}$ (14) \bar{x}, y, z	(3) \bar{y}, x, z (7) $y + \frac{1}{2}, x + \frac{1}{2}, \bar{z}$ (11) $y + \frac{1}{2}, \bar{x} + \frac{1}{2}, \bar{z}$ (15) \bar{y}, \bar{x}, z	(4) y, \bar{x}, z (8) $\bar{y} + \frac{1}{2}, \bar{x} + \frac{1}{2}, \bar{z}$ (12) $\bar{y} + \frac{1}{2}, x + \frac{1}{2}, \bar{z}$ (16) y, x, z	General: $hk: h+k=2n$ $0k: k=2n$ $h0: h=2n$ Special: as above, plus
8 g $\dots m$	x, x, z $\bar{x} + \frac{1}{2}, x + \frac{1}{2}, \bar{z}$	\bar{x}, \bar{x}, z $x + \frac{1}{2}, \bar{x} + \frac{1}{2}, \bar{z}$	\bar{x}, x, z $x + \frac{1}{2}, x + \frac{1}{2}, \bar{z}$	x, \bar{x}, z $\bar{x} + \frac{1}{2}, \bar{x} + \frac{1}{2}, \bar{z}$	no extra conditions
8 f $\dots m$	$0, y, z$ $\frac{1}{2}, y + \frac{1}{2}, \bar{z}$	$0, \bar{y}, z$ $\frac{1}{2}, \bar{y} + \frac{1}{2}, \bar{z}$	$\bar{y}, 0, z$ $y + \frac{1}{2}, \frac{1}{2}, \bar{z}$	$y, 0, z$ $\bar{y} + \frac{1}{2}, \frac{1}{2}, \bar{z}$	no extra conditions
8 e $\dots 2$	$x, x + \frac{1}{2}, 0$ $\bar{x} + \frac{1}{2}, \bar{x}, 0$	$\bar{x}, \bar{x} + \frac{1}{2}, 0$ $x + \frac{1}{2}, x, 0$	$\bar{x} + \frac{1}{2}, x, 0$ $x, \bar{x} + \frac{1}{2}, 0$	$x + \frac{1}{2}, \bar{x}, 0$ $\bar{x}, x + \frac{1}{2}, 0$	no extra conditions
4 d $2mm$	$\frac{1}{2}, 0, z$	$0, \frac{1}{2}, z$	$0, \frac{1}{2}, \bar{z}$	$\frac{1}{2}, 0, \bar{z}$	no extra conditions
4 c $\dots 2/m$	$\frac{1}{4}, \frac{1}{4}, 0$	$\frac{3}{4}, \frac{3}{4}, 0$	$\frac{3}{4}, \frac{1}{4}, 0$	$\frac{1}{4}, \frac{3}{4}, 0$	$hk: h, k = 2n$
2 b $4m$	$0, 0, z$	$\frac{1}{2}, \frac{1}{2}, \bar{z}$			no extra conditions
2 a $\bar{4}m2$	$\frac{1}{2}, 0, 0$	$0, \frac{1}{2}, 0$			no extra conditions

Symmetry of special projections

Along [001] $p4mm$
 $\mathbf{a}' = \frac{1}{2}(\mathbf{a} - \mathbf{b})$ $\mathbf{b}' = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
 Origin at $0, 0, z$

Along [100] $\neq 2mg$
 $\mathbf{a}' = \mathbf{b}$
 Origin at $x, \frac{1}{4}, 0$

Along [110] $\neq 2mm$
 $\mathbf{a}' = \frac{1}{2}(-\mathbf{a} + \mathbf{b})$
 Origin at $x, x, 0$

Maximal non-isotypic subgroups

I	[2] $p\bar{4}m2$ (59)	1; 2; 7; 8; 11; 12; 13; 14
	[2] $p\bar{4}2_1m$ (58)	1; 2; 5; 6; 11; 12; 15; 16
	[2] $p4mm$ (55)	1; 2; 3; 4; 13; 14; 15; 16
	[2] $p4_22$ (54)	1; 2; 3; 4; 5; 6; 7; 8
	[2] $p4/n11$ ($p4/n$, 52)	1; 2; 3; 4; 9; 10; 11; 12
	[2] $p2/n12/m$ ($cmme$, 48)	1; 2; 7; 8; 9; 10; 15; 16
	[2] $p2/n2_1/m1$ ($pmmn$, 46)	1; 2; 5; 6; 9; 10; 13; 14

IIa none

IIb none

Maximal isotypic subgroups of lowest index

IIc [9] $p4/nmm$ ($\mathbf{a}' = 3\mathbf{a}, \mathbf{b}' = 3\mathbf{b}$) (64)

Minimal non-isotypic supergroups

I none

II [2] $c4/mmm$ ($p4/mmm$, 61)

5.2. Guide to the use of the scanning tables

5.2.1. Introduction

The global symmetry of an ideal crystal is described by its space group \mathcal{G} . It is also of interest to consider symmetries of local character. The classical example is that of the site symmetries, which are the symmetries of individual points in a crystal. These are completely described and classified as a part of the standard description of space groups in *International Tables for Crystallography*, Volume A, *Space-Group Symmetry* (IT A, 2005). The results of this procedure contain two types of information:

- (i) site symmetries of individual points under the action of the group \mathcal{G} and
- (ii) orbits of points under the action of the group \mathcal{G} .

This information, apart from its use, for example, in the consideration of the splitting of atomic levels in the field of the site symmetry, provides the background for the description of crystal structure: points of the same orbit are occupied by identical atoms (ions) and the environment of these atoms (ions) is also identical. A complete description of the structure is reduced to a description of the occupation of individual Wyckoff positions.

Analogously, we may consider the symmetries of planes transecting the crystal and of straight lines penetrating the crystal, called here the *sectional layer groups* (*symmetries*) and the *penetration rod groups* (*symmetries*). Here we look again for the two types of information:

- (i) symmetries of individual planes (straight lines) under the action of the group \mathcal{G} and
- (ii) orbits of planes (straight lines) under the action of the group \mathcal{G} .

The general law that describes the connection between local symmetries and orbits of points, planes or straight lines is expressed by a coset resolution of the space group with respect to local symmetries. The orbits of planes (straight lines) have analogous properties to orbits of points. The structure of the plane (straight line) and its environment is identical for different planes (straight lines) of the same orbit. This is useful in the consideration of layer structures, see Section 5.2.5.1, and of structures with pronounced rod arrangements.

Layer symmetries have also been found to be indispensable in *bicrystallography*, see Section 5.2.5.2. This term and the term *bicrystal* were introduced by Pond & Bollmann (1979) with reference to the study of grain boundaries [see also Pond & Vlachavas (1983) and Vlachavas (1985)]. A bicrystal is in general an edifice where two crystals, usually of the same structure but of different orientations, meet at a common boundary – an interface. The sectional layer groups are appropriate for both the description of symmetries of such boundary planes and the description of the bicrystals.

The *sectional layer groups* were, however, introduced much earlier by Holser (1958*a,b*) in connection with the consideration of domain walls and twin boundaries as symmetry groups of planes bisecting the crystal. The mutual orientations of the two components of a bicrystal are in general arbitrary. In the case of domain walls and twin boundaries, which can be considered as

interfaces of special types of bicrystals, there are crystallographic restrictions on these orientations. The group-theoretical basis for an analysis of domain pairs is given by Janovec (1972). The consideration of the structure of domain walls or twin boundaries involves the sectional layer groups (Janovec, 1981; Zikmund, 1984); they were examined in the particular cases of domain structure in KSCN crystals (Janovec *et al.*, 1989) and of domain walls and antiphase boundaries in calomel crystals (Janovec & Zikmund, 1993), see Section 5.2.5.3, and recently also in fullerene C₆₀ (Janovec & Kopský, 1997; Saint-Grégoire *et al.*, 1997).

The first attempts to derive the sectional layer groups systematically were made by Wondratschek (1971) and by using a computer program written by Guigas (1971). Davies & Dirl (1993*a*) developed a program for finding subgroups of space groups, which they later modified to find sectional layer groups and penetration rod groups as well (Davies & Dirl, 1993*b*). The use and determination of sectional layer groups have also been discussed by Janovec *et al.* (1988), Kopský & Litvin (1989) and Fuksa *et al.* (1993).

The penetration rod groups can be used in the consideration of linear edifices in a crystal, *e.g.* line dislocations or intersections of boundaries, or in crystals with pronounced rod arrangements. So far, there seems to be no interest in the penetration rod groups and there is actually no need to produce special tables for these groups. Determining penetration rod groups was found to be a complementary problem to that of determining sectional layer groups (Kopský, 1989*c*, 1990).

The keyword for this part of this volume is the term *scanning*, introduced by Kopský (1990) for the description of the spatial distribution of local symmetries. In this sense, the description of site symmetries and classification of point orbits by Wyckoff positions are a result of the *scanning* of the space group for the site symmetries.

The *Scanning tables*, Part 6, give a complete set of information on the space distribution of sectional layer groups and of the penetration rod groups. They were derived using the *scanning-group* method and the *scanning theorem*, see Section 5.2.2.2. The tables describe explicitly the scanning for the sectional layer groups. The spatial distribution of (scanning for) the penetration rod groups is seen directly from the *scanning groups*, which are given as a part of the information in the scanning tables.

The sectional layer groups and the penetration rod groups are subgroups of space groups and as such act on the three-dimensional point space. The examples of particular studies in Section 5.2.5 emphasize the importance of the exact location of sectional layer groups with reference to the crystal structure and hence to the crystallographic coordinate system. In the usual interpretation, Hermann–Mauguin symbols do not specify the location of the group in space. In the scanning tables, each Hermann–Mauguin symbol means a quite specific space or layer group with reference to a specified crystallographic coordinate system, see Sections 5.2.3.1.1 and 5.2.3.1.4.

The layer and rod groups can also be interpreted as factor groups of reducible space groups (Kopský, 1986, 1988, 1989*a,b*, 1993*a*; Fuksa & Kopský, 1993). Our choice of standard Hermann–

5. SCANNING OF SPACE GROUPS

For the special values $h = 1, l = 0$, this orbit turns into an orbit (110) with fixed parameters and an orthorhombic scanning group.

5.2.5. Applications

5.2.5.1. Layer symmetries in crystal structures

The following two examples show the use of layer symmetries in the description of crystal structures.

Example 1

Fig. 5.2.5.1 shows the crystal structure of cadmium iodide, CdI_2 . The space group of this crystal is $P\bar{3}m1, D_{3d}^3$ (No. 164). The anions form a hexagonal close packing of spheres and the cations occupy half of the octahedral holes, filling one of the alternate layers. In close-packing notation, the CdI_2 structure is:

A	C	B	C
I	Cd	I	void

From the scanning tables, we obtain for planes with the (0001) orientation and at heights $0\mathbf{c}$ or $\frac{1}{2}\mathbf{c}$ a sectional layer symmetry $p\bar{3}m1$ (L72), and for planes of this orientation at any other height a sectional layer symmetry $p3m1$ (L69).

The plane at height $0\mathbf{c}$ contains cadmium ions. This plane defines the orbit of planes of orientation (0001) located at points $P + n\mathbf{c}$, where $n \in Z$ (Z is the set of all integers). All these planes contain cadmium ions in the same arrangement (C layer filled with Cd).

The plane at height $\frac{1}{2}\mathbf{c}$ defines the orbit of planes of orientation (0001) located at points $P + (n + \frac{1}{2})\mathbf{c}$, where $n \in Z$. All these planes lie midway between A and B layers of iodine ions with the B layer below, the A layer above the plane. They contain only voids.

The planes at levels $\frac{1}{4}\mathbf{c}$ and $\frac{3}{4}\mathbf{c}$ contain B and A layers of iodine ions, respectively. These planes and all planes produced by translations $n\mathbf{c}$ from them belong to the same orbit because the operations $\bar{3}$ exchange the A and B layers.

Example 2

The space group of cadmium chloride, CdCl_2 , is $R\bar{3}m, D_{3d}^5$ (No. 166). Fig. 5.2.5.2 shows the structure of CdCl_2 in its triple hexagonal cell. The anions form a cubic close packing of spheres and the cations occupy half of the octahedral holes of each alternate layer. In close-packing notation, the CdCl_2 structure is:

A	C	B	A	C	B	A	C	B	A	C	B
Cl	Cd	Cl	void	Cl	Cd	Cl	void	Cl	Cd	Cl	void

We choose the origin at a cadmium ion and the hexagonal basis vectors \mathbf{a}, \mathbf{b} as shown in Fig. 5.2.5.2. This corresponds to the obverse setting for which the scanning table is given in Part 6. The planes with the (0001) orientation at the heights $0\mathbf{c}, \frac{1}{6}\mathbf{c}, \frac{1}{3}\mathbf{c}, \frac{1}{2}\mathbf{c}, \frac{2}{3}\mathbf{c}$ and $\frac{5}{6}\mathbf{c}$ have a sectional layer group of the type $p\bar{3}m1$ (L72) and at any other height have a sectional layer group of the type $p3m1$ (L69).

The scanning table also specifies the location of the sectional layer groups. The position along the c axis, where the basis vector $\mathbf{c} = \mathbf{d}$ specifies the scanning direction, is given by fractions of \mathbf{d} or by $s\mathbf{d}$ in the case of a general position. At the heights $0\mathbf{c}$ and $\frac{1}{2}\mathbf{c}$, the sectional layer group is the group $p\bar{3}m1$

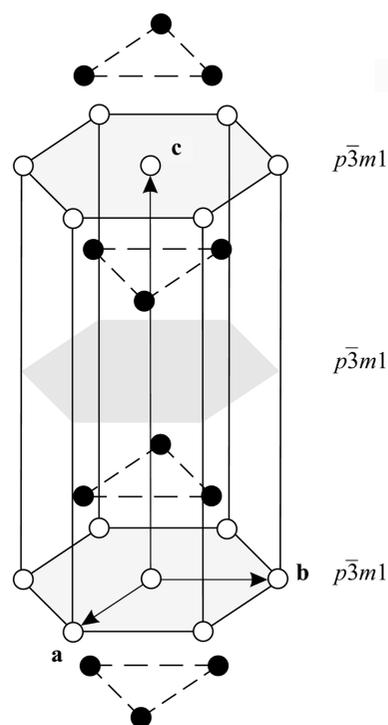


Fig. 5.2.5.1. The structure of cadmium iodide, CdI_2 . The section planes of two orbits in special positions are distinguished by shading. The figure is drastically elongated in the c direction to exhibit the layer symmetries.

(L72), while at the heights $\frac{1}{3}\mathbf{c}$ and $\frac{5}{6}\mathbf{c}$ it is the group $p\bar{3}m1$ [$(\mathbf{a} + 2\mathbf{b})/3$] (L72), and at the heights $\frac{2}{3}\mathbf{c}$ and $\frac{1}{6}\mathbf{c}$ it is the group $p\bar{3}m1$ [$(2\mathbf{a} + \mathbf{b})/3$] (L72), where the vectors in brackets mean the shift of the group $p\bar{3}m1$ in space. The planes at the heights $0\mathbf{d}, \frac{1}{3}\mathbf{d}$ and $\frac{2}{3}\mathbf{d}$ belong to one translation orbit and the layers contain cadmium ions which are shifted relative to each other by the vectors $(\mathbf{a} + 2\mathbf{b})/3$ and $(2\mathbf{a} + \mathbf{b})/3$. The planes at the heights $\frac{1}{2}\mathbf{d}, \frac{5}{6}\mathbf{d}$ and $\frac{1}{6}\mathbf{d}$ contain the voids and are located midway between layers of chlorine ions; they belong to another linear orbit and again are shifted relative to each other by the vectors $(\mathbf{a} + 2\mathbf{b})/3$ and $(2\mathbf{a} + \mathbf{b})/3$.

5.2.5.2. Interfaces in crystalline materials

The scanning for the sectional layer groups is a procedure which finds applications in the theory of *bicrystals* and their *interfaces*. The first of these two terms was introduced in the study of grain boundaries (Pond & Bollmann, 1979; Pond & Vlachavas, 1983; Vlachavas, 1985; Kalonji, 1985). An *ideal bicrystal* is understood to be an aggregate of two semi-infinite crystals of identical structure, meeting at a common planar boundary called the *interface*, where one of the structures, occupying half-space on one side of the interface, is misoriented and/or displaced relative to the other structure occupying the other half-space. The word interface is a synonym for a boundary and interfaces considered here are *homophase interfaces*, in contrast with *heterophase interfaces*, where the two structures are different (Sutton & Balluffi, 1995).

An independent study of domain and twin boundaries (Janovec, 1981; Zikmund, 1984) resulted in a terminology parallel to that of the *bicrystallography*. The basic concept here is the *domain twin*, which is technically a particular case of a bicrystal. In this section, we use the terminology of bicrystals, giving the terminology of domain twins, used in the next section, parenthetically.

No. 13 $P2/c$

C_{2h}^4

$$\mathcal{G} = P12/c1 \quad \text{UNIQUE AXIS } b$$

CELL CHOICE 1

$$\mathcal{G} = P112/a \quad \text{UNIQUE AXIS } c$$

Orientation orbit (<i>hkl</i>)	Conventional basis of the scanning group a' b' d	Scanning group \mathcal{H}	Linear orbit sd	Sectional layer group $\mathcal{L}(\mathbf{sd})$	
UNIQUE AXIS <i>b</i> (010)	c a b	$P112/a$	$0\mathbf{d}, \frac{1}{2}\mathbf{d}$ [$\mathbf{sd}, -\mathbf{sd}$]	$p112/a$ $p112(\mathbf{a}/4)$	L07 L03
UNIQUE AXIS <i>c</i> (001)	a b c				
UNIQUE AXIS <i>b</i> (<i>n0m</i>)	b $n\mathbf{c} - m\mathbf{a}$ $p\mathbf{c} + q\mathbf{a}$				
UNIQUE AXIS <i>c</i> (<i>mn0</i>)	c $n\mathbf{a} - m\mathbf{b}$ $p\mathbf{a} + q\mathbf{b}$ <i>n</i> odd <i>m</i> even <i>q</i> odd <i>m</i> odd <i>q</i> odd	$P2/b11$	$0\mathbf{d}, \frac{1}{2}\mathbf{d}$ [$\mathbf{sd}, -\mathbf{sd}$]	$p2/b11$ $pb11$	L16 L12
		$P2/n11$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$ $[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$ [$\pm\mathbf{sd}, (\pm s + \frac{1}{2})\mathbf{d}$]	$p\bar{1}$ $p211(\mathbf{b}'/4)$ $p1$	L02 L08 L01
	<i>p</i> odd <i>q</i> even	$P2/c11$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$ $[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$ [$\pm\mathbf{sd}, (\pm s + \frac{1}{2})\mathbf{d}$]	$p\bar{1}$ $p211$ $p1$	L02 L08 L01

No. 13 $P2/c$

C_{2h}^4

$$\mathcal{G} = P12/n1 \quad \text{UNIQUE AXIS } b$$

CELL CHOICE 2

$$\mathcal{G} = P112/n \quad \text{UNIQUE AXIS } c$$

Orientation orbit (<i>hkl</i>)	Conventional basis of the scanning group a' b' d	Scanning group \mathcal{H}	Linear orbit sd	Sectional layer group $\mathcal{L}(\mathbf{sd})$	
UNIQUE AXIS <i>b</i> (010)	c a b	$P112/n$	$0\mathbf{d}, \frac{1}{2}\mathbf{d}$ [$\mathbf{sd}, -\mathbf{sd}$]	$p112/n$ $p112[(\mathbf{a} + \mathbf{b})/4]$	L07 L03
UNIQUE AXIS <i>c</i> (001)	a b c				
UNIQUE AXIS <i>b</i> (<i>n0m</i>)	b $n\mathbf{c} - m\mathbf{a}$ $p\mathbf{c} + q\mathbf{a}$				
UNIQUE AXIS <i>c</i> (<i>mn0</i>)	c $n\mathbf{a} - m\mathbf{b}$ $p\mathbf{a} + q\mathbf{b}$ <i>n</i> odd <i>m</i> even <i>p</i> even <i>q</i> odd or <i>n</i> even <i>m</i> odd <i>p</i> odd <i>q</i> even <i>p</i> odd <i>q</i> odd	$P2/n11$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$ $[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$ [$\pm\mathbf{sd}, (\pm s + \frac{1}{2})\mathbf{d}$]	$p\bar{1}$ $p211(\mathbf{b}'/4)$ $p1$	L02 L08 L01
		$P2/c11$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$ $[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$ [$\pm\mathbf{sd}, (\pm s + \frac{1}{2})\mathbf{d}$]	$p\bar{1}$ $p211$ $p1$	L02 L08 L01
	<i>n</i> odd <i>m</i> odd	$P2/b11$	$0\mathbf{d}, \frac{1}{2}\mathbf{d}$ [$\mathbf{sd}, -\mathbf{sd}$]	$p2/b11$ $pb11$	L16 L12

No. 224 $Pn\bar{3}m$ O_h^4

$$\mathcal{G} = P_n^4 \bar{3}_m^2 \text{ origin } 2$$

Orientation orbit (<i>hkl</i>)	Conventional basis of the scanning group			Scanning group \mathcal{H}	Linear orbit $s\mathbf{d}$	Sectional layer group $\mathcal{L}(s\mathbf{d})$	
	\mathbf{a}'	\mathbf{b}'	\mathbf{d}				
(001)	\mathbf{a}	\mathbf{b}	\mathbf{c}	$P4_2/nmm$ (origin 2) ($\mathbf{a}'/2$ or $\mathbf{b}'/2$)	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$	$\widehat{c}mme$	L48
(100)	\mathbf{b}	\mathbf{c}	\mathbf{a}		$[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$	$p\bar{4}2m$ [$(\mathbf{a}' + \mathbf{b}')/4$]	L57
(010)	\mathbf{c}	\mathbf{a}	\mathbf{b}		$[\pm s\mathbf{d}, (\pm s + \frac{1}{2})\mathbf{d}]$	$\widehat{c}mm2$ [$(\mathbf{a}' + \mathbf{b}')/4$]	L26
(110)	\mathbf{c}	$\mathbf{a} - \mathbf{b}$	$\mathbf{a} + \mathbf{b}$	$Abmm$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$	$pbmb$	L38
(011)	\mathbf{a}	$\mathbf{b} - \mathbf{c}$	$\mathbf{b} + \mathbf{c}$		$[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$	$pbmm$ ($\mathbf{b}'/4$)	L40
(101)	\mathbf{b}	$\mathbf{c} - \mathbf{a}$	$\mathbf{c} + \mathbf{a}$		$[\pm s\mathbf{d}, (\pm s + \frac{1}{2})\mathbf{d}]$	$pbm2$ ($\mathbf{b}'/4$)	L24
(1 $\bar{1}$ 0)	\mathbf{c}	$\mathbf{a} + \mathbf{b}$	$\mathbf{b} - \mathbf{a}$	$Abmm$ [($\mathbf{b}' + \mathbf{d}$)/4]	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$	$pbmm$	L40
(0 $\bar{1}$ $\bar{1}$)	\mathbf{a}	$\mathbf{b} + \mathbf{c}$	$\mathbf{c} - \mathbf{b}$		$[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$	$pbmb$ ($\mathbf{b}'/4$)	L38
($\bar{1}$ 01)	\mathbf{b}	$\mathbf{c} + \mathbf{a}$	$\mathbf{a} - \mathbf{c}$		$[\pm s\mathbf{d}, (\pm s + \frac{1}{2})\mathbf{d}]$	$pbm2$	L24
(111)	$\mathbf{a} - \mathbf{b}$	$\mathbf{b} - \mathbf{c}$	$\boldsymbol{\tau}$		With respect to origin at P		
($\bar{1}\bar{1}$ 1)	$\mathbf{b} - \mathbf{a}$	$-\mathbf{b} - \mathbf{c}$	$\boldsymbol{\tau}_3$	With respect to origin at $P + (\mathbf{a} + \mathbf{b})/2$			
($\bar{1}\bar{1}\bar{1}$)	$\mathbf{a} + \mathbf{b}$	$\mathbf{c} - \mathbf{b}$	$\boldsymbol{\tau}_1$	With respect to origin at $P + (\mathbf{b} + \mathbf{c})/2$			
($\bar{1}\bar{1}\bar{1}$)	$-\mathbf{a} - \mathbf{b}$	$\mathbf{b} + \mathbf{c}$	$\boldsymbol{\tau}_2$	With respect to origin at $P + (\mathbf{c} + \mathbf{a})/2$			
				$R\bar{3}m$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}, \frac{1}{3}\mathbf{d}, \parallel \frac{2}{6}\mathbf{d}, \frac{2}{3}\mathbf{d}, \parallel \frac{1}{6}\mathbf{d}]$	$p\bar{3}m1$	L72
						$p\bar{3}m1$ [$(2\mathbf{a}' + \mathbf{b}')/3$]	L72
						$p\bar{3}m1$ [$(\mathbf{a}' + 2\mathbf{b}')/3$]	L72
						$p3m1$	L69

No. 225 $Fm\bar{3}m$ O_h^5

$$\mathcal{G} = F_n^4 \bar{3}_m^2$$

Orientation orbit (<i>hkl</i>)	Conventional basis of the scanning group			Scanning group \mathcal{H}	Linear orbit $s\mathbf{d}$	Sectional layer group $\mathcal{L}(s\mathbf{d})$	
	\mathbf{a}'	\mathbf{b}'	\mathbf{d}				
(001)	$(\mathbf{a} - \mathbf{b})/2$	$(\mathbf{a} + \mathbf{b})/2$	\mathbf{c}	$I4/mmm$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$	$p4/mmm$	L61
(100)	$(\mathbf{b} - \mathbf{c})/2$	$(\mathbf{b} + \mathbf{c})/2$	\mathbf{a}		$[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$	$p4/nmm$	L64
(010)	$(\mathbf{c} - \mathbf{a})/2$	$(\mathbf{c} + \mathbf{a})/2$	\mathbf{b}		$[\pm s\mathbf{d}, (\pm s + \frac{1}{2})\mathbf{d}]$	$p4mm$	L55
(110)	\mathbf{c}	$(\mathbf{a} - \mathbf{b})/2$	$(\mathbf{a} + \mathbf{b})/2$	$Immm$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}]$	$pmmm$	L37
(1 $\bar{1}$ 0)	\mathbf{c}	$(\mathbf{a} + \mathbf{b})/2$	$(\mathbf{b} - \mathbf{a})/2$		$[\frac{1}{4}\mathbf{d}, \frac{3}{4}\mathbf{d}]$	$pmmn$ [$(\mathbf{a}' + \mathbf{b}')/4$]	L46
(011)	\mathbf{a}	$(\mathbf{b} - \mathbf{c})/2$	$(\mathbf{b} + \mathbf{c})/2$		$[\pm s\mathbf{d}, (\pm s + \frac{1}{2})\mathbf{d}]$	$pmm2$	L23
(0 $\bar{1}$ $\bar{1}$)	\mathbf{a}	$(\mathbf{b} + \mathbf{c})/2$	$(\mathbf{c} - \mathbf{b})/2$				
(101)	\mathbf{b}	$(\mathbf{c} - \mathbf{a})/2$	$(\mathbf{c} + \mathbf{a})/2$				
($\bar{1}$ 01)	\mathbf{b}	$(\mathbf{c} + \mathbf{a})/2$	$(\mathbf{a} - \mathbf{c})/2$				
(111)	$(\mathbf{a} - \mathbf{c})/2$	$(\mathbf{b} - \mathbf{a})/2$	$\boldsymbol{\tau}$	$R\bar{3}m$	$[0\mathbf{d}, \frac{1}{2}\mathbf{d}, \frac{1}{3}\mathbf{d}, \parallel \frac{2}{6}\mathbf{d}, \frac{2}{3}\mathbf{d}, \parallel \frac{1}{6}\mathbf{d}]$	$p\bar{3}m1$	L72
($\bar{1}\bar{1}$ 1)	$(-\mathbf{a} - \mathbf{c})/2$	$(\mathbf{a} - \mathbf{b})/2$	$\boldsymbol{\tau}_3$		$p\bar{3}m1$ [$(2\mathbf{a}' + \mathbf{b}')/3$]	L72	
($\bar{1}\bar{1}\bar{1}$)	$(\mathbf{a} + \mathbf{c})/2$	$(-\mathbf{a} - \mathbf{b})/2$	$\boldsymbol{\tau}_1$		$p\bar{3}m1$ [$(\mathbf{a}' + 2\mathbf{b}')/3$]	L72	
($\bar{1}\bar{1}\bar{1}$)	$(\mathbf{c} - \mathbf{a})/2$	$(\mathbf{a} + \mathbf{b})/2$	$\boldsymbol{\tau}_2$		$p3m1$	L69	